MAT 667 (Dynamical Systems)
Homework #5, Due Wednesday, April 19, 2017

1. Let f_a be the logistic map family, with parameter $a \in (0, 4]$, defined by

$$f_a : [0, 1] \to [0, 1]; \; f_a(x) = ax(1 - x)$$

(a) Fix the parameter $a \in [1, 4]$. Let $h : [0, 1] \to [-a/2, a/2]; x \mapsto h(x) = a(x - 1/2)$. Find the formula for $g_a(y)$, where $g_a : [-a/2, a/2] \to [-a/2, a/2]$ is defined by $g_a = h \circ f_a \circ h^{-1}$.

(b) Do the same with $h : [0, 1] \to [-1, 1]; x \mapsto h(x) = 2x - 1$.

2. Consider the map $g_\mu : \mathbb{R} \to \mathbb{R}$ defined by $g_\mu(x) = \mu - x^2$. Draw partial bifurcation diagram showing the fixed points and their stability in the $\mu - x$ plane. Comment on the relevance of problem 1(a). In particular, how does the g_μ family with μ in the neighborhood of $-1/4$ compare with the f_a family with a in the neighborhood of 1?

In the next two problems, let $t : [0, 1] \to [0, 1]$ denote the tent map, defined as

$$t : [0, 1] \to [0, 1]; \; t(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

which, you showed in Homework 3 problem 4, is equivalent to

$$t((0.b_0 b_1 b_2 \ldots)_2) = \begin{cases} (0.b_1 b_2 b_3 \ldots)_2 & \text{if } b_0 = 0 \\ (0.b_1^* b_2^* b_3^* \ldots)_2 & \text{if } b_0 = 1 \end{cases}$$

where $b_i \in \{0, 1\}$ are the bits in the base 2 expansion of x, and $b^* = 1 - b$.

3. Let $t : [0, 1] \to [0, 1]$ be the tent map, and $f : [0, 1] \to [0, 1]$ be defined by $f(x) = 2x(\text{mod}1)$. Show that $t \circ t = t \circ f$. This means that t is a semi-conjugacy of f to t.

4. (a) Recall that if h is a semiconjugacy of f to g, then $g^n(y) = (h \circ f^n)(x)$, where x satisfies $h(x) = y$. Use the results of the previous problem to show that, for any integer $n \ge 1$,

$$t^n((0.b_0 b_1 b_2 \ldots)_2) = \begin{cases} (0.b_n b_{n+1} b_{n+2} b_{n+3} \ldots)_2 & \text{if } b_{n-1} = 0 \\ (0.b_n^* b_{n+1}^* b_{n+2}^* b_{n+3}^* \ldots)_2 & \text{if } b_{n-1} = 1 \end{cases}.$$

Show that you can use either pre-image of $(0.b_0 b_1 b_2 \ldots)_2$ under h and get the same result for t^n.

(b) Use the formula from part (a) to find the binary expansion of the period-3 points of t. Note now much easier it is with this new result than it was in homework 3, problem 5.

Exercise T3.6, and exercises 3.4, 3.7, 3.9 and 3.10.