MAT 665, Differential Equations, Prof. Swift

Homework 1, due Friday, Sept. 1 at the beginning of class.

This is a subset of a final I gave for MAT 239 (Differential Equations) in the past, along with a bonus problem at the end. It starts with a summary of formulas that serves as a very quick review of the course.

First order ODEs

These formulas concern solutions to $y' = f(y,t)$.

A separable ODE can be written $f(y)dy = g(t)dt$. Integrate both sides (adding a constant immediately) and try to solve for y to get an explicit solution $y = \varphi(t)$. An implicit solutions has the form $F(y,t) = c$.

The standard form for a first order linear ODE is $y' + p(t)y = g(t)$. This can be solved using the integrating factor $\mu(t) = \exp(\int p(t)\, dt)$.

There is a technique for solving Exact ODEs, and lots of other techniques.

Linear ODEs of order 2 or higher

These formulas concern solutions to $p_n(t)y^{(n)} + \cdots + p_1(t)y' + p_0(t)y = g(t)$.

The general solution of an nth order linear homogeneous ODE is $y = c_1y_1(t) + \cdots + c_ny_n(t)$, where $\{y_i(t) \mid 1 \leq i \leq n\}$ is a linearly independent set of solutions.

If the ODE is homogeneous ($g = 0$) with constant coefficients, then a real root λ of the characteristic equation corresponds to a solution $y = e^{\lambda t}$ of the ODE.

A complex conjugate pair of roots $\lambda = a \pm ib$ of the characteristic equation corresponds to two solutions $y = e^{at}\cos(bt)$ and $y = e^{at}\sin(bt)$of the ODE.

Repeated roots introduce factors of t to get linearly independent solutions.

For non-homogeneous ODEs ($g(t)$ is not the zero function), then the general solution is $y = y_h + y_p$, where y_h is the generral solution to the associated homogeneous equation, and y_p is a particular solution to the non-homogeneous ODE.

Systems of ODEs

These formula concern the solution to $\mathbf{x}' = A\mathbf{x}$, where A is a 2×2 matrix with constant, real entries. The eigenvalues of A satisfy $\det(\mathrm{A} - \lambda \mathrm{I}) = 0$, and the associated eigenvectors satisfy $A\mathbf{v} = \lambda\mathbf{v}$, or $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

If A has complex eigenvalues $\lambda = a \pm i\, b$, with $b \neq 0$, let $\lambda_1 = a + i\, b$, and find the complex eigenvector $\mathbf{v}_1 = Re(\mathbf{v}_1) + i\, Im(\mathbf{v}_1)$. The general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1e^{at}\big(Re(\mathbf{v}_1)\cos(bt) - Im(\mathbf{v}_1)\sin(bt)\big) + c_2e^{at}\big(Im(\mathbf{v}_1)\cos(bt) + Re(\mathbf{v}_1)\sin(bt)\big).$$

If A has a single eigenvalue λ with multiplcity 2, there are two sub-cases.

(a) If $A = \lambda I$, then the solution is easy: $\mathbf{x}(t) = e^{\lambda t}\mathbf{x}(0)$.

(b) Otherwise, let $\mathbf{v}_2$ be any vector with $(A - \lambda I)\mathbf{v}_2 \neq \mathbf{0}$. Then $\mathbf{v}_1 := (A - \lambda I)\mathbf{v}_2$ is an eigenvector, and the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1e^{\lambda t}\mathbf{v}_1 + c_2e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2).$$

1. Solve the ODE. (You do not need to show your work *on this problem.* Just write down the solution, if you know it.)

(a) $\dfrac{dy}{dt} = 3y$, $y(0) = 2$. (b) $\dfrac{d^2y}{dt^2} = -9y$ (c) $\dfrac{dy}{dt} = 2(y-1)$

(d) $\dfrac{d^3y}{dt^3} = 0$ (e) $\dfrac{d^2y}{dt^2} + t^2\dfrac{dy}{dt} + y^2 = 0$, $y(0) = 0$, $y'(0) = 0$.

2. Find the general solution to $y' + \frac{2}{t}y = t$.

3. (a) Find an explicit solution to the IVP $\dfrac{dy}{dt} = y^3$, $y(0) = 1$.
(b) What is the interval on which the solution to part (a) is defined?
(c) Sketch the solution to part (a).

7. In this problem we will model pollution of a lake. Assume that the volume of the lake is 10^6 cubic meters. Initially the water in the lake is pure, but at time $t = 0$ a trans fat factory starts polluting the lake by pumping in effluent with a concentration of 2 kilograms per cubic meter of "substance F". The effluent is pumped in at a rate of 100 cubic meters per day. There is also a source of pure water entering the lake at a rate of 900 cubic meters per day. The pollutant in the lake is well mixed, and the mixed water drains out of the lake at the rate of 1000 cubic meters per day. (Thus, the volume of the lake stays at a constant 10^6 cubic meters.)

(a) Write down the IVP for $y(t)$, the kilograms of substance F in the lake after the factory has been open t days.
(b) Solve the IVP. (You may solve by inspection.)
(c) How many kilograms of substance F are in the lake in the limit $t \to \infty$?

8. Solve the IVP $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

9. This problem concerns a linear nonhomogeneous ODE $L[y] = g(t)$.
Suppose that:
$y_1(t)$ satisfies the ODE with the initial conditions $y(0) = 1$, $y'(0) = 0$,
$y_2(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 1$, and
$y_3(t)$ satisfies the ODE with the initial conditions $y(0) = 0$, $y'(0) = 0$.
Find the solution to the ODE with the initial conditions $y(0) = 3$, $y'(0) = 4$.
(Your answer will be a linear combination of y_1, y_2, and y_3.)

13. Find the general solution to $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}$. Sketch the phase portrait.

14. Find the general solution to $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{x}$. Sketch the phase portrait.

15. Consider the second order ODE $ay'' + by' + cy = 0$, where a, b, and c are constants.
(a) Assume $a \neq 0$. Let $x_1 = y$ and $x_2 = y'$, and convert the second order ODE into the system $\mathbf{x}' = A\mathbf{x}$.
(b) Show that the characteristic equation of the second order ODE, namely $a\lambda^2 + b\lambda + c = 0$, has the same set of roots as the eigenvalues of the matrix A.