Example 3

"Differential Equations and Dynamical Systems".

Example 3 in section 1.8 of Perko:

```
A = {{2, 1, 0}, {0, 2, 0}, {0, -1, 2}};
MatrixForm[A]
Eigenvalues[A]
```

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

$\lambda =$ {2, 2, 2}

```
Alam = A - 2 IdentityMatrix[3];
MatrixForm[Alam]
MatrixForm[RowReduce[Alam]]
```

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

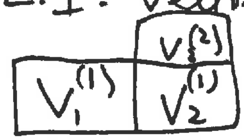
$$\text{rref}(A-2I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \delta_1 = 2$$

```
Alam.Alam
```

$(A-2I)^2$ {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

$\delta_2 = 3$

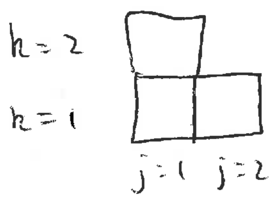
Note: If we choose the size one chain ~~tower~~ first, and then the size 2 ~~tower~~ chain we might not ~~be~~ able to get L.I. vectors

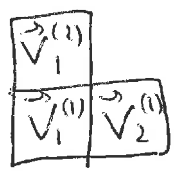


$$V_1^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad V_2^{(2)} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ gives}$$

$$\vec{V}_2^{(1)} = \begin{bmatrix} b \\ 0 \\ -b \end{bmatrix}$$

Any choice of $\vec{V}_2^{(2)}$ makes $\vec{V}_1^{(1)}$ and $V_2^{(1)}$ L.D.

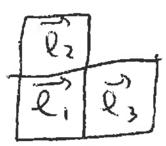




$\in \vec{X}$

$$P = \begin{bmatrix} \vec{V}_1^{(1)} & V_1^{(2)} & V_2^{(1)} \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & \\ 0 & 2 & \\ & & 2 \end{bmatrix}$$



$\in \vec{y}$

$$\vec{V}_1^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ so } \vec{V}_1^{(1)} = (A - 2I)\vec{V}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{V}_2^{(1)} = \vec{0},$$ and $V_2^{(1)}$ is LI from $\vec{V}_1^{(1)}$

I choose $\vec{V}_2^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ so all 3 generalized eigenvectors are orthogonal

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 4 Jim Swift

Example 4 in section 1.8 of Perko:

```
A = {{0, -1, -2, -1}, {1, 2, 1, 1}, {0, 0, 1, 0}, {0, 0, 1, 1}};
MatrixForm[A]
Eigenvalues[A]
```

$\delta_1 = 2, \delta_2 = 4$ gives , or

$\vec{V}_1^{(2)}$	$\vec{V}_2^{(2)}$
$\vec{V}_1^{(1)}$	$\vec{V}_2^{(1)}$

$$A = \begin{pmatrix} 0 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\lambda =$ {1, 1, 1, 1}

```
Alam = A - IdentityMatrix[4];
MatrixForm[Alam]
MatrixForm[RowReduce[Alam]]
```

$$A - \lambda I \quad \begin{pmatrix} -1 & -1 & -2 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{rref}(A - \lambda I) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \delta_1 = \cancel{3}\,2$$

```
Alam = A - IdentityMatrix[4];
MatrixForm[Alam.Alam]
```

$$(A - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \delta_2 = 4$$

Step 2.3

$j=1$: $\vec{V}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ gives $\vec{V}_1^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$j=2$ $\vec{V}_2^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ gives $\vec{V}_2^{(1)} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Note that $\vec{V}_2^{(2)}$ can be any vector, ~~[illegible]~~

$\vec{V}_2^{(2)} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, with $c \neq 0$. This is

because $A - \lambda I \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -a-b-2c-d \\ a+b+c+d \\ 0 \\ c \end{bmatrix}$

This vector is L.D. with $\vec{V}_1^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

if and only if $c = 0$.

So $P = \begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

gives

$$P^{-1}AP = \left[\begin{array}{cc:cc} 1 & 1 & & \\ 0 & 1 & & \\ \hdashline & & 1 & 1 \\ & & 0 & 1 \end{array}\right]$$

Example 5 Jim Swift

Example 5 in section 1.8 of Perko:

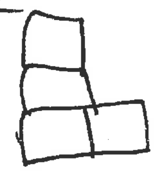
```
A = {{0, -2, -1, -1}, {1, 2, 1, 1}, {0, 1, 1, 0}, {0, 0, 0, 1}};
MatrixForm[A]
Eigenvalues[A]
```

$$A = \begin{pmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\lambda =$ {1, 1, 1, 1} $m = 4$

Step 2.2

3
2
$j=1$



$s_1 = 3 \quad s_2 = 1$

```
Alam = A - IdentityMatrix[4];
MatrixForm[Alam]
MatrixForm[RowReduce[Alam]]
```

$$A - \lambda I = \begin{pmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rref}(A-\lambda I) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_1 = 2$$

Step 2.3

$j=1$. choose $\vec{V}_1^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Then $\vec{V}_1^{(2)} = (A-\lambda I)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

and $\vec{V}_1^{(1)} = (A-\lambda I)^2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

```
MatrixForm[Alam.Alam]
MatrixForm[RowReduce[Alam.Alam]]
```

$$(A-\lambda I)^2 = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{rref}((A-\lambda I)^2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_2 = 3$$

$j=2$. We need an eigenvector with a nonzero 4th component.

I choose $\vec{V}_2^{(1)} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

```
MatrixForm[Alam.Alam.Alam]
```

$$(A-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_4 = 4$$

Then $P = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ gives

$$P^{-1}AP = \left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & \\ \hline & & & 1 \end{array}\right].$$

Example 4 Done Incorrectly.

Example 4 in section 1.8 of Perko:

```
A = {{0, -1, -2, -1}, {1, 2, 1, 1}, {0, 0, 1, 0}, {0, 0, 1, 1}};
MatrixForm[A]
Eigenvalues[A]
```

$$A=\begin{pmatrix} 0 & -1 & -2 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\lambda = \{1, 1, 1, 1\}$ $m=4$

```
Alam = A - IdentityMatrix[4];
MatrixForm[Alam]
MatrixForm[RowReduce[Alam]]
```

$$A-\lambda I=\begin{pmatrix} -1 & -1 & -2 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{rref}(A-\lambda I)=\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_1=2$$

```
Alam = A - IdentityMatrix[4];
MatrixForm[Alam.Alam]
```

$$(A-\lambda I)^2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \delta_2=4$$

Algorithm is pg 82 (chap. 9) of the 1st edition. There is a second edition available, but I don't have it.

The chain diagram is

$\vec{V}_1^{(2)}$	$\vec{V}_2^{(2)}$
$\vec{V}_1^{(1)}$	$\vec{V}_2^{(1)}$

Claim: Bronson's Algorithm, from Matrix Operations Schaum's outline (1st edition) can give a linearly dependent set.

I choose $\vec{V}_1^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so $\vec{V}_1^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

The algorithm says to choose any $\vec{V}_2^{(2)}$ which is linearly independent from $\vec{V}_1^{(2)}$ and $\vec{V}_1^{(1)}$.

I foolishly choose $V_2^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, which gives $\vec{V}_2^{(1)} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

But then $\vec{V}_1^{(1)} = \vec{V}_2^{(1)}$, and the vectors $\{\vec{V}_1^{(1)}, \vec{V}_1^{(2)}, \vec{V}_2^{(1)}, \vec{V}_2^{(2)}\}$ are linearly dependent.