The first three problems are reviews of MAT 239. In each problem \( y \) is a real valued function of \( t \).

1. This problem concerns a nonhomogeneous second order linear ODE \( L[y] = g(t) \). Suppose that:
   - \( y_1(t) \) satisfies the ODE with the initial conditions \( y(0) = 1, y'(0) = 0 \),
   - \( y_2(t) \) satisfies the ODE with the initial conditions \( y(0) = 0, y'(0) = 1 \), and
   - \( y_3(t) \) satisfies the ODE with the initial conditions \( y(0) = 0, y'(0) = 0 \).
   Find the solution to the ODE with the initial conditions \( y(0) = \alpha, y'(0) = \beta \). Justify your answer.
   Note that \( y_1(t) \) and \( y_2(t) \) are two different real-valued functions, they are not two components of the vector \( y(t) \).

2. Solve the ODE. Give the general solution if no initial condition is given, otherwise give the unique solution to the Initial Value Problem (IVP). You may solve these by inspection.
   (a) \( \frac{dy}{dt} = 3y, \ y(0) = 2 \).
   (b) \( \frac{d^2y}{dt^2} = -9y \)
   (c) \( \frac{dy}{dt} = 2(y - 1) \)
   (d) \( \frac{d^3y}{dt^3} = 0 \)
   (e) \( \frac{d^2y}{dt^2} + t^2 \frac{dy}{dt} + y^2 = 0, \ y(0) = 0, \ y'(0) = 0 \).

3. (a) Solve the initial value problem \( \frac{d^2y}{dt^2} - 2y = 0, \ y(0) = 1, \ y'(0) = 0 \).
   (b) Solve the initial value problem \( \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0, \ y(0) = 0, \ y'(0) = 1 \).
   (c) Solve the initial value problem \( \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0, \ y(0) = \alpha, \ y'(0) = \beta \) for arbitrary \( \alpha \) and \( \beta \).

The next two problems concern scaling. Assume that all parameters, e.g. \( m \) and \( \gamma \), are positive.

4. Consider the damped harmonic oscillator:
   \[ m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0 \]
   where \( x(t) \in \mathbb{R} \). This ODE has three parameters. Do a scaling \( x = \alpha \bar{x}, \ t = \beta \bar{t} \), to put this system into the form:
   \[ \frac{d^2\bar{x}}{d\bar{t}^2} + \gamma' \frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0 \].
Find the value of $\gamma$ in terms of the original parameters. If $m$ has units of grams, $\gamma$ has units of grams per second, and $k$ has units of grams per second squared, what are the units of $\gamma$? This problem shows that we can consider oscillations of stars or oscillations of electrons with the same equation. The mass $m$ can be huge or minuscule, but all that matters is the dimensionless friction parameter $\gamma$.

5. In problem 4 you may have noticed that the value of $\alpha$ in the scaling could be anything without effecting the scaled equation. This is because each term is proportional to $x$. In this problem you will scale two equations where the value of $\alpha$ can be chosen to eliminate another parameter in the ODE.

(a) Scale
\[
m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = F \cos(\omega t)
\]
to become
\[
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + x = \cos(\omega t),
\]
and find the values of $\gamma$ and $\omega$ in terms of the original parameters. What is the significance of $x = 1$? Hint: What would happen if the original oscillator were subjected to a constant force of size $F$?

(b) Scale
\[
m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx - cx^3 = 0
\]
to become
\[
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} = x - x^3 = 0
\]
and find the value of $\gamma$ in terms of the original parameters. What is the significance of $x = 1$?

6. Solve the IVP
\[
\dot{x}_1 = -x_1^3 \\
\dot{x}_2 = -x_2
\]
with $x_1(0) = 1$, $x_2(0) = 1$. Show that the trajectory lies on the curve $x_2 = ae^{-b/x_1^2}$ for some choice of $a$ and $b$. Sketch the phase portrait of the 2-dimensional ODE.

Do these problems from the textbook:

§1.1: 1acde, 2ac, 3, 4, 5.

§1.2: 1ac, 2, 3ac, 4, 5, 7.