MAT 661 (Applied Mathematics), Prof. Swift
Proof of Theorem 4.2 in Chapter 2

Set-up: Fix a vector field $\mathbf{V} \in C^1(\Omega, \mathbb{R}^3)$ for $\Omega \subseteq \mathbb{R}^3$ and a space curve $\mathcal{C} = \{\mathbf{r}(t) \mid t \in I\}$ for some open interval I and some $\mathbf{r} \in C^1(I, \Omega)$. Assume $\mathbf{0} \notin \mathbf{V}(\Omega)$ and $\mathbf{0} \notin \mathbf{r}'(I)$. That is, $\mathbf{V}(x, y, z)$ is never $\mathbf{0}$ in Ω, and the velocity $\mathbf{r}'(t)$ is never $\mathbf{0}$ in I. Fix a time $t_0 \in I$ and the point $\mathbf{r}_0 := \mathbf{r}(t_0) \in \mathcal{C}$. We assume that there are two functionally independent first integrals $u_1, u_2 \in C^1(\Omega, \mathbb{R})$.

Theorem 4.2: Suppose $\mathbf{V}$ is not tangent to $\mathcal{C}$ at $\mathbf{r}_0$, *i.e.*, $\mathbf{V}(\mathbf{r}_0)$ is not parallel to $\mathbf{r}'(t_0)$. Then there is a neighborhood Ω_0 of $\mathbf{r}_0$ such that there is a unique integral surface of $\mathbf{V}$ containing $\mathcal{C} \cap \Omega_0$.

Proof (by construction): Let $u_1, u_2 \in C^1(\Omega, \mathbb{R})$ be two functionally independent first integrals of V. Define $U_1, U_2 \in C^1(I, \mathbb{R})$ by $U_i(t) = u_i(\mathbf{r}(t))$ for $i = 1, 2$. We seek a "global" C^1 function $F_g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$F_g(U_1(t), U_2(t)) = 0, \text{ for all } t \in I,$$

Given such an F_g, the "global" integral surface containing $\mathcal{C}$ is

$$S_g = \{(x, y, z) \in \Omega \mid F(u_1(x, y, z), u_2(x, y, z)) = 0\},$$

provided $\nabla F(u_1(x, y, z), u_2(x, y, z)) \neq \mathbf{0}$ in Ω. The fact that S_g is an integral surface follows from Theorem 3.1 and Definition 4.1. The curve $\mathcal{C}$ is contained in S_g by construction, since the equation defining S_g holds for all points in $\mathcal{C}$.

Frequently, as in the homework problems, such a global function F_g is obvious. The remainder of the proof shows that a similar "local" function F always exists, although it cannot always be found as a closed-form expression, and the domain of F is restricted. Consider the derivatives

$$U_i'(t_0) = \nabla u_i(\mathbf{r}_0) \cdot \mathbf{r}'(t_0)$$

for $i = 1, 2$. These two derivatives cannot both be 0, since if they were then $\mathbf{V}$ would be tangent to $\mathcal{C}$ at $\mathbf{r}_0$. Assume, without loss of generality, that $U_1'(t_0) \neq 0$. Then the inverse function theorem says that there is a local inverse function $U_1^{-1} \in C^1(D, \mathbb{R})$ that satisfies $U_1^{-1} \circ U_1(t) = t$ for all t in some open interval I_0 containing t_0. The domain of U_1^{-1} is $D = U_1(I_0)$. Note that $U_1'(t) \neq 0$ for all $t \in I_0$; otherwise the inverse of U_1 restricted to I_0 would not be defined. Therefore V is never tangent to $\mathbf{r}(I_0)$, which is a subset of $\mathcal{C}$. The inverse property implies that $U_2(t) = U_2 \circ U_1^{-1} \circ U_1(t)$ for all $t \in I_0$. Define $f : D \to \mathbb{R}$ by $f = U_2 \circ U_1^{-1}$, and define $F : D \times \mathbb{R} \to \mathbb{R}$ by $F(p, q) = q - f(p)$. Then $F(U_1(t), U_2(t)) = U_2(t) - f(U_1(t)) = 0$ for all $t \in I_0$. Let Ω_0 be a neighborhood of $\mathbf{r}_0$ such that $\mathcal{C} \cap \Omega_0 = \mathbf{r}(I_0)$. Then $\mathcal{C} \cap \Omega_0$ is contained in $S = \{(x, y, z) \in \Omega_0 \mid F(u_1(x, y, z), u_2(x, y, z)) = 0\}$. The gradient, $\nabla F(u_1, u_2) = -f'(u_1)\nabla u_1 + \nabla u_2$, is never $\mathbf{0}$ in Ω_0 since u_1 and u_2 are functionally independent. Thus, S is an integral surface and we are done.

Following the book, we do not prove the uniqueness of S, but the result seems intuitive given the constructive proof. □

A note about the stream function ψ.

Theorem 5.1 in the book states that every *solenoidal*, also known as *incompressible* or *divergence-free*, vector field $\mathbf{V} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ can be written as $\mathbf{V} = \nabla \times \mathbf{W}$, for some "vector potential" $\mathbf{W} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. The identity $\nabla \cdot (\nabla \times \mathbf{W}) = 0$ proves that such a $\mathbf{V}$ is incompressible, but it takes the machinery developed in this chapter to prove that such a $\mathbf{W}$ always exists. In physics we usually use $\mathbf{A}$ instead of $\mathbf{W}$, especially for the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, which is famously solenoidal since there are no magnetic monopoles. (A solenoid is a coil of wire used to create magnetic fields.) The role of the vector potential $\mathbf{A}$ is especially important in Quantum Mechanics.

The vector potential has its drawbacks, though. It is not unique, and we are simply trading one vector field for another. (Both require 3 real-valued functions.) The special case where the vector potential is most useful is for vector fields with no z-component, and where the x and y components do not depend on z. Consider $\mathbf{V} : \Omega \to \mathbb{R}^3$ of the form $\mathbf{V}(x, y, z) = (u(x, y), v(x, y), 0)$. Assume that $\mathbf{V}$ is incompressible, so $\nabla \cdot \mathbf{V} = u_x + v_y = 0$ on Ω. Then we can write $\mathbf{V}$ in terms of a single "stream function" $\psi : \Omega \to \mathbb{R}$ as $\mathbf{V} = \nabla \times (0, 0, \psi) = (\psi_y, -\psi_x, 0)$.
Given certain smoothness assumptions, the stream function ψ exists and is unique up to an additive constant. Finding ψ is similar to solving exact first order ODEs. I like to write the two equations

$$\psi_y(x, y) = u(x, y), \quad \psi_x = -v(x, y)$$

and do two integrals and get "constants" of integration that depend on the other variable.
The book problem 5.5(b) has you find the stream function for

$$\mathbf{V}(x, y, z) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, 0 \right)$$

on $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\}$. This is the electric field outside of a charged wire. I have added a problem to have you find the stream function for the magnetic field caused by a current in the same wire:

$$\mathbf{B}(x, y, z) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right).$$

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