MAT 661 (Applied Mathematics), Prof. Swift
Homework # 3 Comments

From Homework 3, several people had trouble with problem 1.9. This problem is a good example of the notation used in ODEs that Nandor does not approve of. Writing $x = x(t)$ is problematic. Is x a variable or is x a function? We usually use $x(t)$ to indicated the function, even though strictly $x : \mathbb{R} \to \mathbb{R}$ is the function and $x(t) \in \mathbb{R}$.

same integral curves in Ω.

1.9. **Let C be an integral curve of the vector field $\mathbf{V} = (P, Q, R)$ and suppose that C is given parametrically by the equations,**

$$x = x(t), \qquad y = y(t), \qquad z = z(t); \qquad t \in I$$

where the functions $x(t)$, $y(t)$, $z(t)$ are in $C^1(I)$ and the tangent vector

$$\mathbf{T}(t) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right)$$

never vanishes for $t \in I$.

(a) Show that there is a function $\mu(t)$ in $C^1(I)$ such that for every $t \in I$, $\mu(t) \neq 0$ and

$$\mathbf{V}(x(t), y(t), z(t)) = \mu(t)\mathbf{T}(t).$$

(b) Let $t = t(\tau)$ be a solution of the differential equation

$$\frac{dt}{d\tau} = \mu(t)$$

where τ varies over some interval I' as t varies over I. Set

$$\bar{x}(\tau) = x(t(\tau)), \ \bar{y}(\tau) = y(t(\tau)), \ \bar{z}(\tau) = z(t(\tau)).$$

Show that in terms of the new parametric representation

$$x = \bar{x}(\tau), \qquad y = \bar{y}(\tau), \qquad z = \bar{z}(\tau); \qquad \tau \in I',$$

the curve C is a solution curve of the system of equations associated with $\mathbf{V}$,

$$\frac{dx}{d\tau} = P, \qquad \frac{dy}{d\tau} = Q, \qquad \frac{dz}{d\tau} = R.$$

1.9 (a) Assume that $\mathcal{C}$ is an integral curve of $\mathbf{V} = (P, Q, R)$, given parametrically by

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t \in I.$$

Since $\mathcal{C}$ is an integral curve, the tangent vector $\mathbf{T}(t) = (x'(t), y'(t), z'(t))$ is parallel to the vector field $\mathbf{V}$ at the point $(x(t), y(t), z(t))$, and both vectors are nonzero. This translates to the equation

$$\mathbf{V}(x(t), y(t), z(t)) = \mu(t)\mathbf{T}(t)$$

where $\mu \in C^1(I, \mathbb{R})$ is never 0 on I. The smoothness of μ follows from the smoothness assumptions on $\mathbf{T}$ and the parametrization of $\mathcal{C}$.

1.9 (b). Assume that $t(\tau)$ is a solution to the differential equation

$$\frac{dt}{d\tau} = \mu(t),$$

which means that the function $t \in C^1(I', I)$ satisfies $t'(\tau) = \mu(t(\tau))$.
Now, define $\tilde{x}(\tau) = x(t(\tau))$, and similarly define $\tilde{y}$ and $\tilde{z}$. If we parameterize the curve $\mathcal{C}$ with parameter τ we get

$$x = \tilde{x}(\tau), \quad y = \tilde{y}(\tau), \quad z = \tilde{z}(\tau), \quad \tau \in I'.$$

The chain rule says

$$\tilde{x}'(\tau) = x'(t(\tau))t'(\tau) = x'(t(\tau))\mu(t(\tau)),$$

and the other two functions, $\tilde{y}$ and $\tilde{z}$, have similar derivatives. Thus

$$\frac{d}{d\tau}(\tilde{x}, \tilde{y}, \tilde{z}) = \mu(t(\tau))\mathbf{T}(t(\tau)) = \mathbf{V}(x(t(\tau)), y(t(\tau)), z(t(\tau)) = \mathbf{V}(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{z}(\tau)).$$

Since $\mathbf{V} = (P, Q, R)$, we see that $\tilde{x}(\tau)$ satisfies $\tilde{x}'(\tau) = P(\tilde{x}(\tau), \tilde{y}(\tau), \tilde{z}(\tau))$, and $\tilde{y}(\tau)$ and $\tilde{z}(\tau)$ satisfy similar equations. In the notation of ODEs, we say that $x = \tilde{x}(\tau)$, $y = \tilde{y}(\tau)$, and $z = \tilde{z}(\tau)$ satisfy the sytem of three ODEs

$$\frac{dx}{d\tau} = P(x, y, z), \quad \frac{dy}{d\tau} = Q(x, y, z), \quad \frac{dz}{d\tau} = R(x, y, z).$$

This ends the proof, but some interpretation is in order. The definition of an integral curve only says that $\mathbf{V}$ is tangent to $\mathcal{C}$. The parameterizaton of the curve is not part of the definition. The solutions to the usual differential equations $\frac{dx}{dt} = P$, etc., or equivalently $\mathbf{T} = \mathbf{V}$, give a special parameter, t, for the curve. That special parameter in this problem is τ, not the original parameter t.