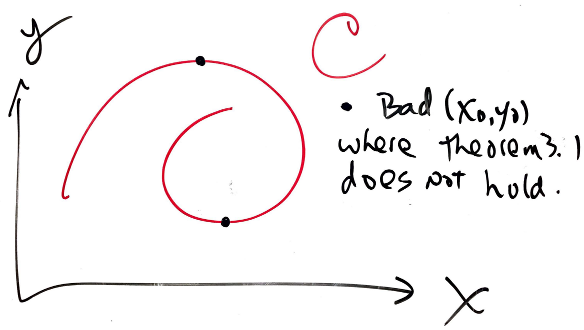
MAT 661 (Applied Mathematics), Prof. Jim Swift @ NAU
Examples of Existence, Non-Existence and Non-Uniqueness of Solutions to an IVP for a first order PDE

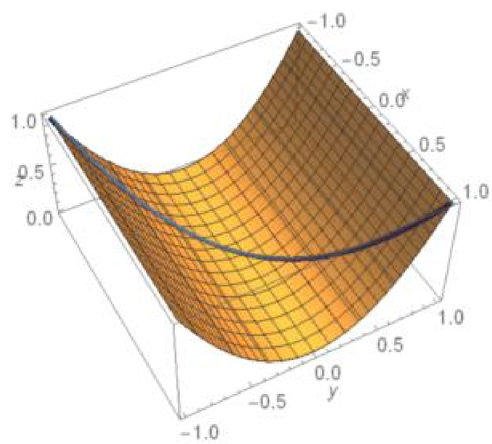
Consider the PDE $z_x = 0$, so $\mathbf{V}(x, y, z) = (1, 0, 0)$. The integral curves of $\mathbf{V}$ are lines parallel to $(1, 0, 0)$, and the general integral of $u_x = 0$ is $u(x, y, z) = F(y, z)$.

This whole document will consider this same PDE with different initial curves $\mathcal{C} \subseteq \mathbb{R}^2$ and initial data $\mathcal{C}' = \mathbf{r}(I) \subseteq \mathbb{R}^3$. Assume $\mathbf{r}(t_0) = \mathbf{r}_0 = (x_0, y_0, z_0)$. We are concerned with existence and uniqueness of solutions $z(x, y)$ in a neighborhood of (x_0, y_0).

Note that $\mathbf{V} \times \mathbf{r}'(t) = (0, -\hat{\mathbf{k}} \cdot \mathbf{r}'(t), \hat{\mathbf{j}} \cdot \mathbf{r}'(t))$. Thus, if $\hat{\mathbf{j}} \cdot \mathbf{r}'(t_0) \neq 0$, then Theorem 3.1 holds, and there is a unique solution in some neighborhood of (x_0, y_0). The condition $\hat{\mathbf{j}} \cdot \mathbf{r}'(t_0) \neq 0$ means that the curve $\mathcal{C}$ in the $x - y$ plane does not have a *horizontal* tangent. (The y component of the velocity of a point tracing out $\mathcal{C}$ is nonzero.) I drew this incorrectly in class.

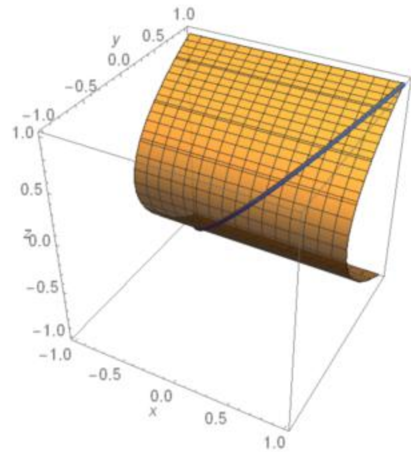


Example 0: (Theorem 3.1 Holds, Unique Solution.), Let $\mathcal{C}$ be the line $y = x$ and let $\mathbf{r}(t) = (t, t, t^2)$. Theorem 3.1 holds everywhere on $\mathcal{C}$, and the unique solution is $z = y^2$. The graph of the solution is yellow, and the curve $\mathcal{C}'$ is shown in blue.

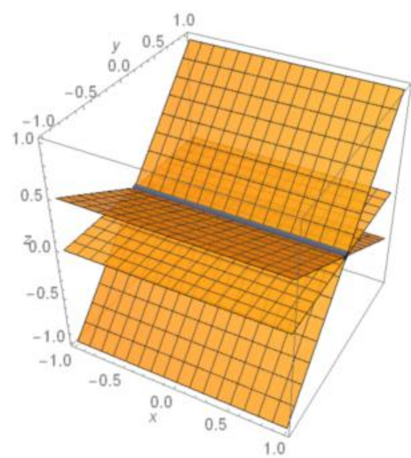


Now, consider various initial value problems where $\mathbf{r}(0) = (x_0, y_0, z_0) = (0, 0, 0)$ and $\hat{\mathbf{j}} \cdot \mathbf{r}'(0) = 0$. That is, $\mathcal{C}$ has a horizontal tangent at $(0, 0)$. The main purpose of this document is to consider examples with these "bad points" where Theorem 3.1 does not hold. I called them "Case" I, II , and III in class, but they are more properly called examples, as I do here. I have also added a few bonus examples (IV and V) that I did not do in class.

Example I: (Theorem 4.1 Holds, No Solutions.) Let $\mathbf{r}(t) = (t, t^2, f(t))$. The inital curve is $\mathcal{C} : y = x^2$ which has a horizontal tangent at $(0, 0)$. We find that $\mathbf{V} \times \mathbf{r}'(t) = (0, -f'(0), 0)$, so if $f'(0) \neq 0$ then Theorem 4.1 says that there is no solution to the IVP in any neighborhood of $(0, 0)$. An example is $\mathbf{r}(t) = (t, t^2, t)$. In this case $u(x, y, z) = y - z^2 = 0$ is the unique integral surface of $\mathbf{V}$ containing $\mathcal{C}'$, but this cannot be solved for z as a function of x and y in any neighborhood of $(0, 0)$.



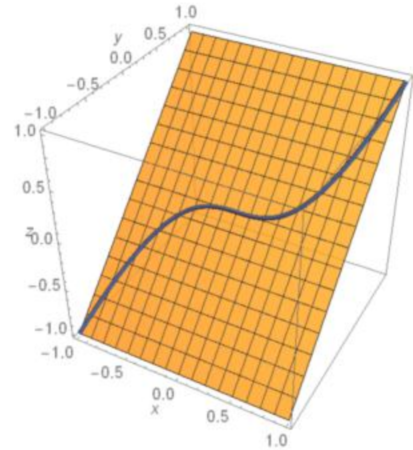
Example II: (Theorem 4.2 Holds: Infinitely Many Solutions.) Let $\mathbf{r}(t) = (t, 0, 0)$, so $\mathbf{V}(\mathbf{r}(t)) = \mathbf{r}'(t) = (1, 0, 0)$ and Theorem 4.2 holds. In this case $\mathcal{C}'$ is the x-axis, which is an integral curve of $\mathbf{V}$, and there are infinitely many solutions, of the form $z(x, y) = f(y)$ for any $f \in C^1(\mathbb{R}, \mathbb{R})$.



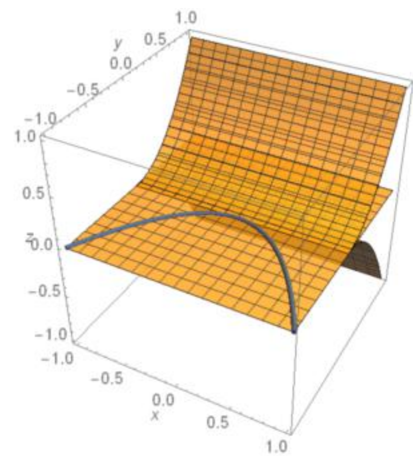
It should be noted that these theorems leave a huge gap. In the cases where $\mathcal{C}'$ is tangent to $\mathbf{V}$ at $\mathbf{r}_0$, but $\mathcal{C}'$ is not an integral curve of $\mathbf{V}$, none of Theorems 3.1, 4.1 or 4.2 hold.

This document ends with 3 examples where $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}'(0) = (1, 0, 0)$, so $\mathbf{V} \times \mathbf{r}'(0) = \mathbf{0}$ and Theorems 3.1 and 4.1 do not hold. Furthermore, in these examples $\mathcal{C}'$ is not an integral curve, so Theorem 4.2 does not hold. Example III was done in class, but Examples IV and V are new. I cannot think of an example that has exactly 2 solutions. This is the basis for my conjecture that the Initial Value Problem has either no solutions, a unique solution, or infinitely many solutions (0, 1, or ∞).

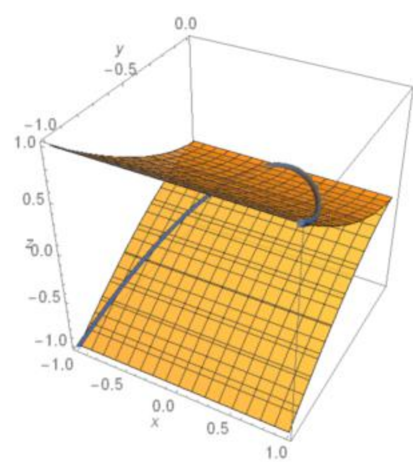
Example III: (None of our Theorems hold, Unique Solution.) Let $\mathbf{r}(t) = (t, t^3, t^3)$. The unique solution is $z = y$, as shown in the figure.



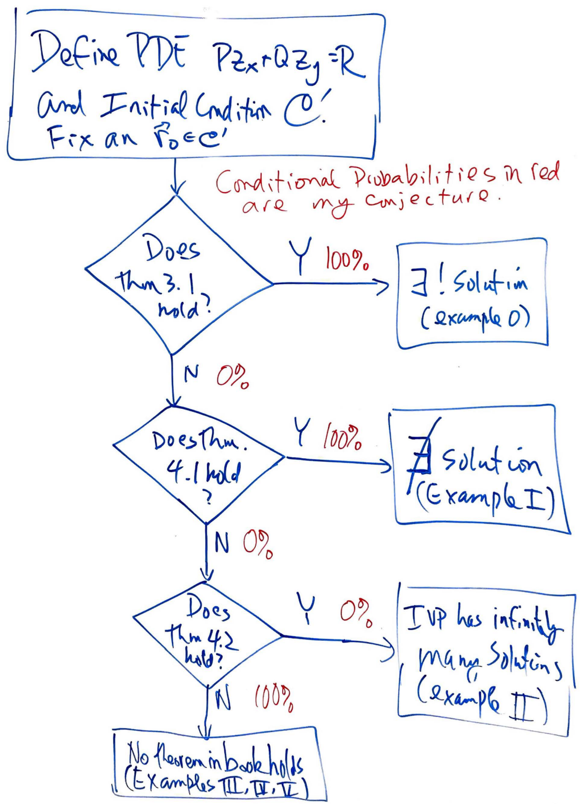
Example IV: (None of our Theorems Hold, Infinitely Many Solutions.) Let $\mathbf{r}(t) = (t, -t^2, 0)$. The integral surface is $z = 0$ for $y \leq 0$, but there is no restriction for $y > 0$ except that the solution must be continuously differentiable. One family of solutions, with parameter α, is $z(x, y) = 0$ if $y \leq 0$ and $z = \alpha y^2$ if $y > 0$. Graphs of three such solutions are shown in the figure.



Example V: (None of our Theorems Hold, No Solutions.) Let $\mathbf{r}(t) = (t, -t^2, t^3)$. There is a non-smooth solution to $u_x = 0$ that contains $\mathcal{C}'$, namely $u(x, y, z) = y^3 + z^2 = 0$. This does not define a solution $z(x, y)$ to the IVP.



Addendum: This is a flow chart showing how the vaious theorems in the book get sorted out. I have included the conjectured "probability" of various things in red. This is based on my intuition and not theorems. Note that 0% probability does not mean that something is impossible. For example, the probability is 0% that a rational number is chosen from the uniform distribution on the interval $[0, 1]$.



The next figure shows the continuation of the flow chart. Again, conjectures are in red. I also conjecture that Theorem 4.2 infinitely less likely to hold than there being a unique solution when Theorem 3.1 does not hold (the bottom middle box).