MAT 461 (PDEs) Prof. Swift

Midterm 2 will be on Wednesday, November 13, 2013

Section 2.5.2, 2.5.4, Chapter 3, Sections 4.1-4.4 and 12.1-12.3

This exam is open book. I will bring 2 copies of the book for those who don't have paper textbooks. You may bring one page of notes with handwiting on both sides. You may use any formula in the book with out justifying it, if you reference the equation number.

Here is an exam from a previous semester. The solutions will be posted to BbLearn.

1(a). Find the Fourier Sine Series (FSS) of $f(x) = x^2$ on the interval $[0, \pi]$.

Hint: You may use your calculator, or the formula

$$\int x^2 \sin(x)\, dx = (2 - x^2)\cos(x) + 2x\sin(x) + C.$$

For full credit, you should evaluate the trig. functions in your expression for the Fourier coefficients.

1(b). Note: You should NOT use the result of part(a) for part (b). These questions ask for information that you would show on a graph.

What number do you get if your evaluate the FSS of x^2 on $[0, \pi]$ at $x = -2$?

What number do you get if your evaluate the FSS of x^2 on $[0, \pi]$ at $x = \pi$?

What number do you get if your evaluate the FSS of x^2 on $[0, \pi]$ at $x = 27\pi + 1$?

2. In this problem you will find the Fourier Sine Series of $f(x) = x(1 - x)/4$ on $[0, 1]$ without doing the integral. Instead, write down the FSS of f with arbitrary coefficients, differentiate it twice, and use the known FSS of 1 on $[0, 1]$, namely

$$1 \sim \sum_{n=1}^{\infty} \frac{2(1 - (-1)^n)}{n\pi} \sin(n\pi x).$$

3(a). Suppose a string with tension T has a point mass m at position $x = c$ in the interior of the string. Use "F = ma" to find the equations satisfied by the displacement u and its derivative $\frac{\partial u}{\partial x}$ at $x = c$. Assume that $\frac{\partial u}{\partial x}$ is small. Use the book's notation $u(c^+, t)$ to stand for $\lim_{x \to c^+} u(x, t)$.

3(b). Without actually solving the eigenvalue problem, sketch the first three eigenfunctions of a string with uniform density except for a point mass at the midpoint of the string. Assume the "guitar string" boundary conditions that $u = 0$ at the two ends. Comment on whether each natural frequency is smaller, the same, or larger than, the corresponding natural frequency of a string without the point mass.

4. Consider the 3-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

for the space outside a ball of radius a (that is, $\rho > a$). Assume that the solution is spherically symmetric, so u is a function of ρ and t alone. In problem 12.3.6 we found that the general solution is

$$u(\rho, t) = \frac{1}{\rho} F(\rho - ct) + \frac{1}{\rho} G(\rho + ct).$$

Find the solution which satisfies the PDE and the boundary condition

$$u(a, t) = f(t), \text{ for all } t$$

and which consists only of the outgoing spherically symmetric wave. (In physics, we often make the "causality" assumption that there is no incoming spherical wave.) Your final answer should be written in terms of the function f.

5. Suppose $u(x, y)$ solves Laplace's equation $\nabla^2 u = 0$ inside a disk or radius a, and the value of u on the boundary of the disk is given by $u(a, \theta) = 10 - \theta^2$ for $-\pi < \theta \leq \pi$. What is the value of u at the center of the disk ($r = 0$)?