Euler's Method on a Graphing Calculator
## by Jim Swift @ NAU

Euler's method is a way to find approximate solutions to an Initial Value Problem (IVP), which is a differential equation with an initial condition. The slope field applet linked to on our web site uses Euler's method or other more accurate methods to compute approximate solutions to an IVP. This is useful for drawing the pictures, but also because often we cannot find a closed form solution. For example, the IVP

$$\frac{dy}{dx} = x^2 - y^2, \quad y(0) = 1, \quad 0 \le x \le 1$$

does not have a solution that can be written in terms of sines and cosines, exponential and logs, powers and roots, etc.

First we will do Euler's method graphically, then numerically using our calculators. The solution to the IVP is a function, and we will get an approximate graph of the function for $0 \le x \le 1$. Euler's method always needs a *step size*, which is called h. We will start with $h = 0.25$. We will get approximate values of $y(h)$, $y(2h)$, $y(3h)$ and $y(4h) = y(1)$ using Euler's method.

The graph goes through the point $(0, 1)$ so put a dot there. Draw a line segment with the indicated slope between $x = 0$ and $x = 0.25$. Put a dot the the right endpoint. Then use the slope at that dot to draw a line segment between $x = .25$ and $x = .5$. Continue in this way until you get to $x = 1$.
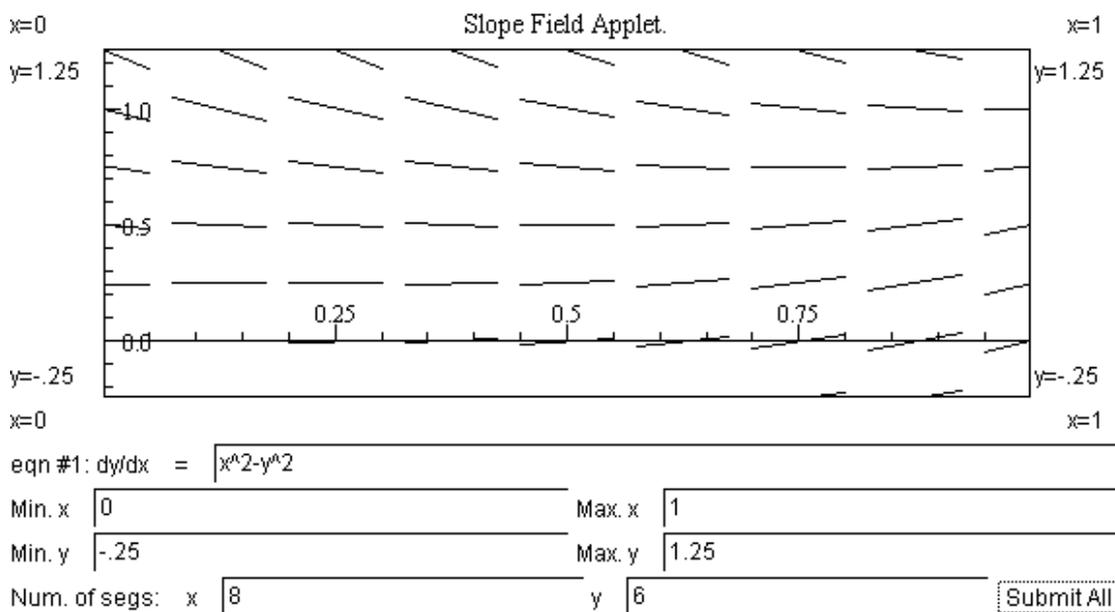
The graph above is a screen shot of Marek Rychlik's JOde, a Java applet for studying Ordinary differential equations, available on the web.

Go to the JOde site and open the first applet on "slope fields". Use the default ODE. Play with the applet for a while, then make the changes shown on the last three lines of the graph above. Click "submit all" and you should see a slope field like the graph above, except that your graph will be more square.

Now, tell JOde to do Euler's method, which is the default method. Set step to 0.25. Enter the initial condition $y(0) = 1$ on the "Add init. cond." line, and click "submit". You should see the line segments that you drew on the slope field above.

Now do the same calculation numerically using your calculator. Start with $x_0 = 0$ and $y_0 = 1$. We know from the start that $x_1 = h = .25$, $x_2 = .5$, $x_3 = .75$ and $x_4 = 1$. Now compute y_1 and fill in the blank:

$$y_1 = y_0 + h * (x_0^2 - y_0^2) = 1 + .25 * (0^2 - 1^2) = \underline{\hspace{1cm}}$$

Now that you know y_1 you can compute y_2 with the formula

$$y_2 = y_1 + h * (x_1^2 - y_1^2) = y_1 + .25 * (.25^2 - y_1^2) = \underline{\hspace{1cm}}$$

Continue in this way to fill in the blanks:

$$y_3 = y_2 + h * (x_2^2 - y_2^2) = y_2 + .25 * (.5^2 - y_2^2) = \underline{\hspace{1cm}}$$

$$y_4 = y_3 + h * (x_3^2 - y_3^2) = y_3 + .25 * (.75^2 - y_3^2) = \underline{\hspace{1cm}}$$

You should find that $y_4 \approx 0.6435$. This means that you have found that $y(1) \approx 0.6435$ on an approximate solution obtained with Euler's method with a step size of $h = 0.25$.

Euler's method is easy to implement on a calculator, especially if you realize that you don't need to keep track of the i in (x_i, y_i). *You should learn how to do this because I may ask you to do Euler's method, with up to 20 steps, on the exam.*

Euler's method to approximate solution to $y' = f(x, y)$, $y(a) = y_0$, $a \le x \le b$:

- Choose step size $h = (b - a)/n$, where n is a positive integer.

- Initialize: Set $x = a$, $y = y_0$. (Note that $x_0 = a$, $x_i = a + ih$, and $x_n = b$.)

- Repeat n times: Replace y with $y + h * f(x, y)$, then replace x with $x + h$.

Here is how to do the previous calculation automatically on a TI-83. The procedure is similar for other calculators. First, initialize the variables. (The $\to$ stands for "STO". The "X" variable has its own key. Use the "ALPHA" key to get Y.)

$$0 \to X : 1 \to Y \text{ ENTER}$$

The colon allows multiple commands. Note that only the output of the last command shows up on the screen. Now do the Euler step with $h = .25$:

$$Y + .25 * (X^2 - Y^2) \to Y : X + .25 \to X : \{X, Y\} \text{ ENTER}$$

You should see "{.25 .75}" on your calculator. You have used the trick of making the last command print out the value of X and Y. Now comes the really fun part: Just press ENTER again! and again, and again until you get the output "{1 .6435 }." This means that $y(1) \approx 0.6435$ with a step size of $h = 0.25$. Now try it again with a step size of $h = 0.125$. Here's a hint: Press 2nd ENTER a few times and you can get back to the commands you previously entered. Go back to where you initialized x, and y. Enter that line, and then go back with 2nd ENTER to enter the *same* command you used before to do the steps, except changing .25 to .125. You should find $y(1) \approx 0.7003$ with a step size of 0.125.

Finally, approximate $y(1)$ for this IVP with a step size of $h = 0.05$. You should find that $y(1) \approx 0.7309$. This is about 2.6% off from $y(1) = 0.75002...$. Euler's method is not very accurate. That is why the slope field applet has other methods besides Euler's method that it can use.