"Theorem 5.3 & 5.4"

Suppose that $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on $D$ and that there is a constant $L > 0$ such that

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \le L \text{ for all } (t, y) \in D.$$

Then, the IVP

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

has a unique solution.

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Example

(1) $y' = |y|$. $a$, and $b$ can be anything.

$$f(t, y) = |y|. \quad \frac{\partial f}{\partial y}(t, y) = \begin{cases} 1 & \text{if } y > 0 \\ \text{undefined} & \text{if } y = 0 \\ -1 & \text{if } y < 0. \end{cases}$$

$f$ is continuous on $D$, but the second hypothesis is false, since $\frac{\partial f}{\partial y}(t, 0)$ is undefined.

The theorem says nothing. However, theorem 5.3 and 5.4 separately (in the book) do show that the IVP has a unique solution.

Example (2) $y' = \begin{cases} y^2 \sin(\frac{1}{y}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$

$f(t,y) = y^2 \sin(\frac{1}{y})$

$\frac{\partial f}{\partial y}(t,y) = 2y \sin(\frac{1}{y}) + y^2 \cos(\frac{1}{y}) \cdot \frac{-1}{y^2}$

$= 2y \sin(\frac{1}{y}) - \cos(\frac{1}{y})$ , if $y \neq 0$

$\frac{\partial f}{\partial y}(t,0) = \lim_{y \to 0} \frac{f(t,y) - f(t,0)}{y}$

$= \lim_{y \to 0} \frac{y^2 \sin(\frac{1}{y})}{y}$

$= \lim_{y \to 0} y \sin(\frac{1}{y})$

$= 0$

Consider $a = 0$, $b > 0$ arbitrary.

"Theorem 5.3 & 5.4" holds, because $f$ is continuous and $\left|\frac{\partial f}{\partial y}(t,y)\right| \leq 2m + 1$ for $|y| \leq m$, and $\lim_{y \to \pm\infty} \left|\frac{\partial f}{\partial y}\right| = 1$. Thus, $\left|\frac{\partial f}{\partial y}\right|$ is bounded.

However $\frac{\partial f}{\partial y}(t,y)$ is not continuous, since

$\lim_{y \to 0} \frac{\partial f}{\partial y} = \lim_{y \to 0} \left[2y \sin(\frac{1}{y}) - \cos(\frac{1}{y})\right]$ DNE.