MAT 239 (Differential Equations), Prof. Swift
The Method of Undetermined Coefficients
§3.5, §4.3 and WeBWorK set 12_linear_nonhomogeneous

The general solution to a *nonhomogeneous* ODE $L[y(t)] = g(t)$ is $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ is the general solution to the *associated homogeneous* ODE $L[y(t)] = 0$, and $y_p(t)$, also called $Y(t)$, is *any* particular solution to the nonhomogeneous ODE.

The method of undetermined coefficients works when $L[y]$ has constant coefficients, and when $g(t)$ involves sums and products of polynomials, exponentials, sines, and cosines. The *form* of the particular solution $y_p(t)$ involves *undetermined coefficients* A, B, C, etc. To find a particular solution, plug $y_p(t)$ into the ODE and figure out the value of the undetermined coefficients.

Some comments about notation: The book uses $Y(t)$ for the particular solution, whereas WeBWorK uses y_p. Also, the general solution to the associated homogeneous equation is sometimes denoted by y_c (the complementary solution) instead of y_h.

The table on p. 181 for the form of y_p is summarized in these two rules, which are justified by the method of annihilators (see pg. 237-238).

Rule 1. The original form of y_p is the general solution of the simplest linear homogeneous ODE with constant coefficients that has $g(t)$ as a solution. Use A, B, C, etc. instead of the constants c_1, c_2, c_3, etc.

Rule 2. If necessary, multiply the original form for y_p by t^s, the smallest power of t such that no terms in the new $y_p(t)$ are also in $y_h(t)$.

Example 1. Find a particular solution to $(D + 1)(D + 2)y = e^t$.

First of all, the characteristic equation is $(k+1)(k+2) = 0$, so $y_h(t) = c_1e^{-t}+c_2e^{-2t}$. Note that $g(t) = e^t$ is a solution to $(D - 1)y = 0$. The operator $(D - 1)$ is called the annihilator of $g(t) = e^t$. The general solution to $(D - 1)y = 0$ is $y = c_1e^t$. Switch c_1 to A and get $y_p(t) = Ae^t$. This "original form" of y_p does not need to be modified since e^t is not a function in y_h for any choice of c_1 and c_2. Plugging this form for y_p into the original nonhomogeneous ODE gives $6Ae^t = e^t$, so $A = \frac{1}{6}$, and $y_p(t) = \frac{1}{6}e^t$. The often painful calculation of A, B, etc. will be left out from now on.

Example 2. Find the form of the particular solution to $(D^2 + 3D + 2)y = \sin(t)$.

The left hand side is the same as for example 1, except it is FOILed out. Thus, $y_h(t) = c_1e^{-t} + c_2e^{-2t}$. Since $g(t) = \sin(t)$ is a solution to $(D^2 + 1)y = 0$, the form of the particular solution is $y_p(t) = A\cos(t) + B\sin(t)$. Rule 2 does not change it.

Example 3. Find the form of y_p for $y'' + 3y' + 2y = e^t + \sin(t)$.

This is the same left-hand side as before. The form of the particular solution is the sum of the previous two (with the letters shifted): $y_p(t) = Ae^t + B\cos(t) + C\sin(t)$.

Example 4. Find the form of y_p for $y'' + 3y' + 2y = t^2 + 3t$.

As before, $y_h(t) = c_1e^{-t}+c_2e^{-2t}$. The right-hand side, $g(t) = t^2+3t$, is annihilated by D^3, and the general solution to $D^3y = 0$ is $y = c_1 + c_2t + c_3t^2$. Rule 2 does not apply, and the form of the particular solution to the nonhomogeneous ODE is $y_p(t) = At^2 + Bt + C$. We traditionally write decreasing powers of t in y_p.

Example 5. Find the form of y_p for $(D+1)(D+2)y = e^{-t}$.

As before, $y_h(t) = c_1e^{-t} + c_2e^{-2t}$. Here $(D+1)$ annihilates $g(t) = e^{-t}$. The original form of the particular solution is $y_p = Ae^{-t}$. Rule 2 comes into effect, and we have to multiply the original form by t to get $y_p(t) = Ate^{-t}$.

Here is why it works: Any solution to $L[y] = (D+1)(D+2)y = e^{-t}$ must satisfy the homogeneous linear ODE $(D+1)^2(D+2)y = (D+1)e^{-t} = 0$. Thus, any solution to $L[y] = e^{-t}$ must be in the family of functions $y = c_1e^{-t} + c_2te^{-t} + c_3e^{-2t}$. But $L[c_1e^{-t} + c_3e^{-2t}] = 0$, so we might as well choose $y_p(t) = c_2te^{-t}$, or equivalently $y_p(t) = Ate^{-t}$.

Examples with $k = \pm 2i$: $y_h(t) = c_1\cos(2t) + c_2\sin(2t)$

$(D^2+4)y = \sin(t)$, $\quad y_p(t) = A\cos(t) + B\sin(t)$. No resonance.

$y'' + 4y = \sin(2t)$, $\quad y_p(t) = t(A\cos(2t) + B\sin(2t))$. Resonance.

$y'' + 4y = e^{-t}\sin(2t)$, $\quad y_p(t) = e^{-t}(A\cos(2t) + B\sin(2t))$. No resonance.

$y'' + 4y = (3t+1)\sin(2t)$, $\quad y_p(t) = t\left[(At+B)\cos(2t) + (Ct+D)\sin(2t)\right]$.

In this last example, avoid this common mistake: $y_p(t) \neq t(At+B)(C\cos(2t) + D\sin(2t))$. You need a separate polynomial in front of the cosine and sine terms.

Practice:

Write down $y_h(t)$ and the form of the particular solution y_p for these ODEs. The linear operator L is given in factored form to make it easier. This *will* be on the test. A pdf of this handout with the solutions is at our website.

$D^2y = t$

$y_h = c_1 + c_2t \qquad y_p = t^2(At+B)$

$(D+1)(D-1)y = e^t + 3e^{2t}$

$y_h = c_1e^{-t} + c_2e^t \qquad y_p = tAe^t + Be^{2t}$

$(D-1)^3y = t^2e^t$

$y_h = (c_1 + c_2t + c_3t^2)e^t \qquad y_p = t^3(At^2 + Bt + C)e^t$

$(D^2+9)^2y = \cos(3t)$

$y_h = (c_1 + c_2t)\cos(3t) + (c_3 + c_4t)\sin(3t) \qquad y_p = t^2(A\cos(3t) + B\sin(3t))$