# Euler's Method on a Graphing Calculator
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Euler's method is a way to find approximate solutions to an Initial Value Problem (IVP). The slope field applet written by Darryl Nester (linked to on our web site) uses Euler's method or other more accurate methods to compute approximate solutions to an IVP. This is useful for drawing the pictures, but also because often we cannot find a closed form solution. For example, the IVP

$$\frac{dy}{dx} = x^2 - y^2, \quad y(0) = 1, \quad 0 \le x \le 1$$

does not have a solution that can be written in terms of sines and cosines, exponential and logs, powers and roots, etc.
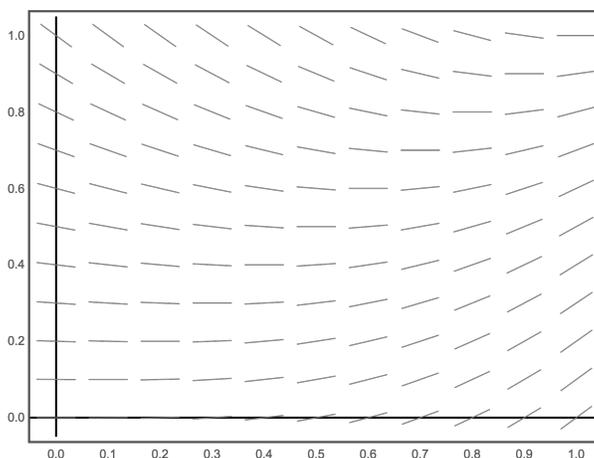
Euler's method is intimately related to the slope field. To start out, put a dot at the initial condition $(x_0, y_0) = (0, 1)$. Sketch the solution to the ODE that goes through that point. At every point in the graph of the solution, the slope of the solution curve is the slope of the line segment that would be there if there were a line segment at every point.

Now we will do Euler's method, first graphically then numerically using our calculators. Euler's method always needs a *step size*, which is called h, but could be called Δx. We will start with $h = 0.5$ to get approximate values of $y(0.5)$, and $y(1)$.

Starting at the initial point $(x_0, y_0) = (0, 1)$, use an ID card to draw a line segment with the indicated slope between $x = 0$ and $x = 0.5$. The slope is $0^2 - 1^2 = -1$, so $\Delta y = h * (-1) = -0.5$. Put a dot at the right endpoint, which is $(x_1, y_1) = (0.5, 0.5)$. The slope at that new point is $.5^2 - .5^2 = 0$. Use the slope at that dot to draw a horizontal line segment between $(0.5, 0.5)$ and $(x_2, y_2) = (1, 0.5)$.

That is a poor approximation of the solution! We estimate that $y(1) = 0.5$ using Euler's method with $h = 0.5$. To get a better approximation, do Euler's method graphically again, this time with step size $h = 0.25$. The starting point is $(x_0, y_0) = (0, 1)$ as before. This time, $\Delta y = h * (-1) = -.25$. The first point is $(x_1, y_1) = (0.25, 0.75)$. Since there is no slope line at this new point, you have to interpolate the nearby slopes to get to (x_2, y_2). Continue in this way until you get to $x_4 = 1$.

Now do the same calculation numerically using your calculator. Start with $x_0 = 0$ and $y_0 = 1$. We know from the start that $x_1 = h = .25$, $x_2 = .5$, $x_3 = .75$ and $x_4 = 1$. Now compute y_1 and fill in the blank:

$$y_1 = y_0 + h * (x_0^2 - y_0^2) = 1 + .25 * (0^2 - 1^2) = \underline{}$$

Now that you know y_1 you can compute y_2 with the formula

$$y_2 = y_1 + h * (x_1^2 - y_1^2) = y_1 + .25 * (.25^2 - y_1^2) = \underline{}$$

Continue in this way to fill in the blanks:

$$y_3 = y_2 + h * (x_2^2 - y_2^2) = y_2 + .25 * (.5^2 - y_2^2) = \underline{}$$

$$y_4 = y_3 + h * (x_3^2 - y_3^2) = y_3 + .25 * (.75^2 - y_3^2) = \underline{}$$

You should find that $y_4 \approx 0.6435$. This means that you have found that $y(1) \approx 0.6435$ on an approximate solution using Euler's method with a step size of $h = 0.25$.

Euler's method is easy to implement on a calculator, especially if you realize that you don't need to keep track of the n in (x_n, y_n). You will need this for WeBWorK, and I may ask you to do Euler's method, with up to 20 steps, on the exam.

Euler's method to solve $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, $x_0 \leq x \leq x_f$ with step size h is:

- Initialize: Set $x = x_0$, $y = y_0$. (You may want to initialize the variable h.)

- Repeat until $x = x_f$:

    Replace y with $y + h * f(x, y)$,

    Replace x with $x + h$.

Here is how to do it on a TI-83. The procedure is similar for other calculators. First, initialize the variables. (The $\rightarrow$ stands for "STO". The "X" variable has its own key. Use the "ALPHA" key to get Y.)

$$0 \rightarrow X : 1 \rightarrow Y \text{ ENTER}$$

The colon allows multiple commands. Note that only the output of the last command shows up on the screen. Now do the Euler step

$$Y + .25 * (X^2 - Y^2) \rightarrow Y : X + .25 \rightarrow X : \{X, Y\} \text{ ENTER}$$

You should see "{.25 .75}" on your calculator. You have used the trick of making the last command print out the value of X and Y. Now comes the really fun part: Press ENTER again! and again, and again until you get the output "{1 .6435 }." This means that $y(1) \approx 0.6435$ with a step size of $h = 0.25$. Now try it again with a step size of $h = 0.125$. Here's a hint: Press 2nd ENTRY a few times and you can get back to the commands you previously entered. Go back to where you initialized x, and y. Enter that line, and then go back with 2nd ENTRY to enter the *same* command you used before to do the steps, except changing .25 to .125. You should find $y(1) \approx 0.7003$ with a step size of 0.125.

Finally, approximate $y(1)$ for this IVP with a step size of $h = 0.05$. You should find that $y(1) \approx 0.7309$. This is about 2.6% off from the true value of $y(1) = 0.75002....$ Even with this small step size, Euler's method is not very accurate. That is why the slope field applet has other methods besides Euler's method that it can use.