Theory of Linear Homogeneous ODEs
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Consider the Linear Homogeneous ODE

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Suppose that p and q are continuous on an open interval I, possibly $I = (-\infty, \infty)$, and $t_0 \in I$, and $y_1(t)$ and $y_2(t)$ are solutions to $L[y] = 0$ on I. Then the following are equivalent. (That means that either they are all true, or they are all false.)

- y_1 and y_2 are linearly independent functions on I. That is, $\Big(c_1\, y_1(t) + c_2\, y_2(t) = 0 \text{ for all } t\Big) \Rightarrow \Big(c_1 = c_2 = 0\Big)$
- $\{y_1(t), y_2(t)\}$ is a Fundamental Solution Set. (This is the *definition* of FSS.)
- The general solution to the ODE on I is $y = c_1\, y_1(t) + c_2\, y_2(t)$.
- The family of functions $y = c_1\, y_1(t) + c_2\, y_2(t)$ can be used to solve every IVP $L[y] = 0$, $y(t_0) = y_0$, $y'(t_0) = v_0$ (where y_0, v_0 are any constants).
- The Wronskian evaluated at t_0 is nonzero. That is, $W(y_1, y_2)(t_0) \neq 0$.
- The Wronskian is never 0 on I. That is, $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

This generalizes. For an nth order Linear Homogeneous ODE, a fundamental solution set has n linearly independent solutions, and the Wronskian is

$$W(y_1, y_2, \cdots y_n)(t) = \det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ {y_1}'(t) & {y_2}'(t) & \cdots & {y_n}'(t) \\ {y_1}''(t) & {y_2}''(t) & \cdots & {y_n}''(t) \\ \cdots & \cdots & \cdots & \cdots \\ {y_1}^{(n-1)}(t) & {y_2}^{(n-1)}(t) & \cdots & {y_n}^{(n-1)}(t) \end{bmatrix}.$$

The importance of the Wronskian follows from Abel's Theorem, and this fact:
Suppose M is a square $(n \times n)$ matrix, and $\mathbf{b}$ is a constant vector. Consider the equation $M\mathbf{x} = \mathbf{b}$ for the unknown $\mathbf{x}$. (This can be written as a system of n linear equations in the n unknowns $x_1, x_2, \ldots, x_n$.)

- If $\det(M) \neq 0$, then the equation has a unque solution, namely $\mathbf{x} = M^{-1}\mathbf{b}$.
- If $\det(M) = 0$, then the equation has either no solutions or an infinite number of solutions.

Some facts about any square matrix M.

- If $\det(M) \neq 0$, then rows of M are a linearly independent set of vectors. The columns are linearly independent too. Furthermore, M has an inverse, M^{-1}.
- If $\det(M) = 0$, then rows of M are a linearly dependent set of vectors. The columns are linearly dependent too. Furthermore, M does not have an inverse.