MAT 239 (Differential Equations)
Theory of Linear Homogeneous Second Order ODEs (§3.2,3)

Topics:
General solutions, fundamental solution sets, linearly independent functions, and the Wronskian.

The topic of discussion is all linear homogeneous second order ODEs
\[ y'' + p(t)y' + q(t)y = 0 \]  
(1)

For simplicity, I assume in these notes that \( p(t) \) and \( q(t) \) are continuous for all \( t \). The book handles the more general case.

These two sections in the book are not about how to find solutions to equation (1). Suppose you have found, by hook or by crook (or by characteristic equation) two solutions, \( y_1 \) and \( y_2 \), to equation (1). These sections answer this question: Is the linear combination
\[ y = C_1y_1(t) + C_2y_2(t) \]  
(2)
the general solution to equation (1)? By definition, the family of functions (2) is the general solution of (1) if \( C_1 \) and \( C_2 \) can be chosen to satisfy any initial conditions \( y(t_0) = \alpha \) and \( y'(t_0) = \beta \). In this case, \( y_1 \) and \( y_2 \) are called a fundamental solution set of equation (1).

The complicating factor is that many different choices of \( y_1 \) and \( y_2 \) are fundamental solution sets. Two different looking families of functions can both be general solutions, since they both contain the same functions. For example, the ODE
\[ y'' - y = 0 \]  
(3)
has the characteristic equation \( r^2 - 1 = 0 \). The roots are \( r_1 = 1 \) and \( r_2 = -1 \), so \( y_1 = e^t \) and \( y_2 = e^{-t} \) are two solutions. “The” general solution, according to the algorithm of §3.1, is
\[ y = C_1e^t + C_2e^{-t} \]  
(4)
On the other hand, if we apply Theorem 3.2.5 to equation (3), we find that \( y_1 = 1/2(e^t + e^{-t}) = \cosh(t) \) and \( y_2 = 1/2(e^t - e^{-t}) = \sinh(t) \). Therefore, equation (3) also has “the” general solution
\[ y = C_1 \cosh(t) + C_2 \sinh(t) \]  
(5)

To solve a particular initial value problem, the values of \( C_1 \) and \( C_2 \) in equation (4) are different from the values of \( C_1 \) and \( C_2 \) in equation (5). For example, the solution to (3) with \( y(0) = 7 \) and \( y'(0) = 3 \) is \( y = 5e^t + 2e^{-t} = 7 \cosh(t) + 3 \sinh(t) \).

An important observation is that we cannot use any two solutions and have (2) be the general solution. For example, \( y_1(t) = 0 \) for all \( t \) is a solution to (1). So is \( y_2(t) = 0 \) for all \( t \). Certainly we know that \( y(t) = C_1 \cdot 0 + C_2 \cdot 0 = 0 \) is not the general solution! Furthermore, if we choose two solutions that are constant multiples of each other, for
example $y_1(t)$ and $y_2(t) = 5y_1(t)$, then $y = C_1y_1(t) + C_2y_2(t) = (C_1 + 5C_2)y_1(t)$ is not the general solution.

Now you are probably saying, “what a stupid thing to do.” You’re right. Of course you would choose two nonzero functions which are not multiples of each other. In other words, you would choose two \textit{linearly independent} functions.

\textbf{Definition:} Two functions $y_1(t)$ and $y_2(t)$ are \textit{linearly dependent} if there are two constants $C_1$ and $C_2$, not both zero, such that $C_1y_1(t) + C_2y_2(t) = 0$ for all $t$. Otherwise, $y_1$ and $y_2$ are \textit{linearly independent}.

\textbf{Theorem:} Two functions $y_1(t)$ and $y_2(t)$ are \textit{linearly independent} if

\[ C_1 y_1(t) + C_2 y_2(t) = 0 \text{ for all } t \]

Otherwise, the functions are \textit{linearly dependent}.

\textbf{Remark:} The definition and the theorem are equivalent. The theorem is perhaps easier to use. The expression $P$ implies $Q$ is \textit{false} if it is possible for $P$ to be true while $Q$ is false. Apply this to the theorem: two functions are linearly dependent if we can choose $C_1$ and $C_2$ such that “$C_1 y_1(t) + C_2 y_2(t) = 0$ for all $t$” is true even though “$C_1 = C_2 = 0$” is false. We see that the theorem and the definition are saying the same thing.

\textbf{Examples:}

(1) Let $y_1(t) = t$ and $y_2(t) = t^2$. Assume that $C_1 t + C_2 t^2 = 0$ for all $t$. At $t = 1$ we find that $C_1 + C_2 = 0$. At $t = -1$ we find that $-C_1 + C_2 = 0$. Therefore $C_1 = C_2 = 0$. By definition, $y_1(t) = t$ and $y_2(t) = t^2$ are linearly independent.

(2) Suppose $y_1(t) = 0$. Then $y_1$ and any function $y_2(t)$ are linearly dependent since $C_1 = 1$ and $C_2 = 0$ will make the linear combination identically 0.

(3) Suppose $y_2(t) = 5y_1(t)$. Then $y_1$ and $y_2$ are linearly dependent, since $C_1 = 5$ and $C_2 = -1$ will make the linear combination $C_1 y_1 + C_2 y_2 = 5y_1 - 5y_1 = 0$ for all $t$.

The essential results in these two sections can be stated quite compactly:

\textbf{Theorem:} Suppose $p(t)$ and $q(t)$ in equation (1) are continuous. Suppose that $y_1(t)$ and $y_2(t)$ are two solutions to (1). Then there are only two possibilities:

\textit{either the following are all true:}
- $y = C_1y_1(t) + C_2y_2(t)$ is the general solution to (1), and
- $y_1$ and $y_2$ are a fundamental solution set of (1), and
- $y_1$ and $y_2$ are linearly independent, and
- $W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is never equal to 0,

\textit{or the following are all true:}
- $y = C_1y_1(t) + C_2y_2(t)$ is \textit{not} the general solution to (1), and
- $y_1$ and $y_2$ are \textit{not} a fundamental solution set of (1), and
- $y_1$ and $y_2$ are linearly dependent, and
- $W(y_1, y_2)(t) = 0$ for all $t$
To be honest, the theory in these two sections is a sledgehammer used to crack a peanut. But the payoff is that all of the theorems generalize to Linear Homogeneous ODEs of any order, which are tougher nuts to crack.