MAT 239, Differential Equations, Prof. Swift

The Final Exam will be Wednesday, Dec. 10, 2003, from 12:30 to 2:30

This is a copy of a previous semester’s final exam. Three pages of notes with writing on both sides are allowed. A graphing calculator is expected. There are no difficult calculations or hard integrals if you use the correct approach. If you have a question, ask me. Show your work!

1. Find the general solution to $y' + y = e^{-t} \sin(t)$.

2. Match the first order ODE to the description. Put a letter in the blank. Hint: Use the differential form of the ODE to test exactness.

   A: $y' = \frac{y^2}{\sin t}$, or $\sin t \; dy - y^2 dt = 0$.

   B: $y' = y \sin t$, or $dy - y \sin t \; dt = 0$

   C: $y' = t^2 - y^2$, or $dy - (t^2 - y^2) dt = 0$

   D: $y' = t + y$, or $dy - (t + y) dt = 0$

   E: $y' = \frac{1 + y^2 \sin t}{2y \cos t}$, or $2y \cos t \; dy - (1 + y^2 \sin t) dt = 0$

   ____ Linear homogeneous
   ____ Linear nonhomogeneous
   ____ Nonlinear and separable
   ____ Nonlinear and exact but not separable
   ____ None of the above

3. (a) Which of the ODEs mentioned in problem 2 has the following slope field? ____
   (b) On the slope field, sketch the solution to this ODE with the initial condition $y(1) = 0$. 

   ![Slope Field]
4. Suppose that the general solution to a linear ODE is \( y(t) = c_1 e^t + c_2 e^{-t} + c_3 + e^{2t} \).

Is the ODE homogeneous or nonhomogeneous?

What is the order of the ODE?

Suppose that the general solution to a linear ODE is \( y(t) = c_1 e^t + c_2 e^{-t} + e^t \).

Is the ODE homogeneous or nonhomogeneous?

Find the linear ODE whose general solution is \( y(t) = c_1 e^t + c_2 e^{-t} + e^{2t} \).

5. Find the form of a particular solution to the ODE, but do not compute the undetermined coefficients.

(a) \( y'' + y = t \cos t \)

(b) \( y''' + y' = 5 + \sin t \)

(c) \( y'' + 6y' + 9y = e^{-3t} \)

6. Find the recurrence relation for the series solution to \( y'' + xy' + y = 0 \).

7. Find the solution to \( x^2y'' + xy' + y = 0, \ y(1) = 0, \ y'(1) = 1 \).

Hint: This is an Euler equation. Look for solutions of the form \( y = x^r \). Euler’s formula is \( e^{ix} = \cos(x) + i \sin(x) \).

8. A second order ODE with a regular singular point at \( x = 0 \) has the indicial equation \( r^2 - 1 = 0 \). Thus, the roots of the indicial equation are \( \pm 1 \).

How many solutions of the form \( y = x^r \sum_{n=0}^{\infty} a_n x^n \) do you expect?

The recurrence relation is \( a_1 = 0 \) and \( a_n = \frac{a_{n-2}}{1 - (n + r)^2} \) for \( n = 2, 3 \cdots \). Find the first three nonzero terms in the solution with \( r = 1 \) and \( a_0 = 1 \).
9. Find the general solution to \( x' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} x \).

Hint: The eigenvalues are \( r_1 = 1, r_2 = 2, \) and \( r_3 = 3. \)

Here are some bonus problems. These problems cover topics that the previous exam did not have time for.

B1. Find an explicit solution to the IVP \( y' = 2ty^2, \) \( y(0) = 1. \) Sketch the solution. Indicate the interval on which the solution is defined.

B2. Determine which of these two ODEs has a regular singular point at \( x = 0. \) Find the indicial equation for this ODE.

(A) \( xy'' + y' + xy = 0 \)

(B) \( x^2y'' + y' + xy = 0 \)

B3. Determine if the vectors are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

\[
\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\\
\]

B4. Consider the system of autonomous nonlinear 1st order ODEs

\[
dx/dt = y, \quad dy/dt = x - x^3
\]

(a) Find all critical points (equilibrium points) of this system.
(b) Find the single 1st order ODE for \( dy/dx \) satisfied by the trajectories.
(c) Show that the ODE from part (b) can be written in a differential form that is exact, and find an equation of the form \( H(x, y) = C \) satisfied by the trajectories.

Solutions: 1. \( y = e^{-t}(-\cos(t) + c). \) 2. BDAEC 3. C 4. nonhomogeneous, 3, homogeneous, \( y'' - y = 3e^{2t}. \) 5. \( a_n = t[(At + B) \cos t + (Ct + D) \sin t] \)
(b) \( Y = At + t[B \cos t + C \sin t] \) (c) \( Y = t^2e^{-3t}. \) 6. \( a_{n+2} = -a_n/(n + 2) \) for \( n = 0, 1, 2, 3, \ldots \) 7. \( y = \sin(\ln x). \) 8. One. \( y = x(1 - 1/8x^2 + 1/192x^4 - \cdots). \) 9. \( x = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t}. \) B1. \( y = 1/(1 - t^2). \) The solution is defined on \((-1, 1). \) B2. A has a regular singular point, and the indicial equation is \( r^2 = 0. \) B3. They are linearly dependent. A relation is \( 2x^{(1)} - x^{(2)} + x^{(3)} = 0. \) B4. (a) \((-1, 0), (0, 0), \) and \((1, 0). \) (b) \( dy/dx = (x - x^3)/y. \) (c) \( ydy - (x - x^3)dx = 0, \)
\[ y^2/2 - (x^2/2 - x^4/4) = c. \]