Homework Due Monday, Sept. 29
Read §3.4, 3.5
Do §3.4 # 2, 4, 6, 7, 8, 17, 21, 26, 29
Integrate $\int e^t \sin(2t) dt$ and $\int e^{\lambda t} \cos(\mu t) dt$ using the technique described in this handout.

Today’s Topic:
How to solve ODEs like $y'' + y = 0$, whose characteristic equation has complex conjugate roots.

Complex Numbers: If $a$ and $b$ are real numbers, then $a + i b$ is a complex number. The real part of $a + i b$ is $a$. The imaginary part of $a + i b$ is $b$, not $i b$. Note that the imaginary part of a complex number is a real number! We use the notation $\text{Re}(a + i b) = a$ and $\text{Im}(a + i b) = b$.

Complex numbers are used all the time in math, physics and engineering. It is a good idea for you to get very comfortable with them.

The rules of arithmetic are quite easy. I will use $a$, $b$, $c$, and $d$ for real numbers and $z$ and $w$ for complex numbers:

\[
(a + i b) + (c + i d) = (a + c) + i(b + d), \quad \text{or} \quad \text{Re}(z + w) = \text{Re}(z) + \text{Re}(w) \quad \text{and} \quad \text{Im}(z + w) = \text{Im}(z) + \text{Im}(w)
\]

\[
(a + i b)(c + i d) = (ac - bd) + i(ad + bc), \quad \text{or} \quad \text{Re}(zw) = \text{Re}(z)\text{Re}(w) - \text{Im}(z)\text{Im}(w) \quad \text{and} \quad \text{Im}(zw) = \text{Im}(z)\text{Re}(w) + \text{Re}(z)\text{Im}(w)
\]

A complex valued function can be written $y(t) = u(t) + i v(t)$, and then $\text{Re}(y(t)) = u(t)$ and $\text{Im}(y(t)) = v(t)$. Euler’s formula is an example of a complex valued function:

\[
e^{it} = \cos(t) + i \sin(t)
\]

The book gives a justification of Euler’s formula in terms of the Maclaurin expansion of the exponential, sine, and cosine functions. The usual rules of algebra and calculus hold for complex exponentials. For example

\[
e^{z+w} = e^z e^w, \quad \text{and} \quad \frac{d}{dt} e^{y(t)} = e^{y(t)} y'(t)
\]

when $z$ and $w$ are complex numbers, and when $y(t)$ is a complex valued function of $t$. Therefore, $(e^{rt})' = re^{rt}$ holds even if $r$ is a complex constant. If $r = \lambda + i \mu$, where $\lambda$ and $\mu$ are real, we have

\[
e^{rt} = e^{(\lambda + i \mu)t} = e^{\lambda t} e^{i \mu t} = e^{\lambda t} (\cos(\mu t) + i \sin(\mu t)) = e^{\lambda t} \cos(\mu t) + i e^{\lambda t} \sin(\mu t)
\]
**Linear Operators:** Recall that a linear homogeneous second order ODE can be written in terms of a linear operator $L$:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Since $L$ is linear, it operates on complex functions with the rule

$$L[u(t) + iv(t)] = L[u(t)] + iL[v(t)]$$

Therefore, if $y(t)$ is a complex valued function that satisfies a linear homogeneous ODE, then its real and imaginary parts, Re($y(t)$) and Im($y(t)$), are real solutions that satisfy the ODE.

**Example:** Find the general solution to $y'' + 2y' + 17y = 0$. The characteristic equation is $r^2 + 2r + 17 = 0$, which has roots $r = -1 \pm 4i$. Therefore $y_1(t) = e^{(-1+4i)t}$ and $y_2(t) = e^{(-1-4i)t}$ are both complex valued solutions. Two real solutions are the real and imaginary parts of $y_1$:

$$u(t) = \text{Re}(e^{(-1+4i)t}) = e^{-t} \cos(4t) \text{ and } v(t) = \text{Im}(e^{(-1+4i)t}) = e^{-t} \sin(4t)$$

The general solution is $y = C_1u(t) + C_2v(t) = C_1e^{-t} \cos(4t) + C_2e^{-t} \sin(4t)$.

**Today’s Rule:** Consider a linear homogeneous ODEs with constant coefficients. If the characteristic equation has the roots $\lambda \pm i \mu$, with $\mu \neq 0$, then this pair of roots corresponds to the pair of real solutions $u(t) = e^{\lambda t} \cos(\mu t)$ and $v(t) = e^{\lambda t} \sin(\mu t)$. (These solutions are linearly independent, provided $\mu \neq 0$, since $W(u, v)(t) = e^{2\lambda t} \mu$.)

**Integrals:** Another use of linear operators and complex valued functions involves the integral, which satisfies

$$\int (u(t) + iv(t))dt = \int u(t)dt + i \int v(t)dt$$

$$\int \text{Re}(z(t))dt = \text{Re}\left(\int z(t)dt\right)$$

$$\int \text{Im}(z(t))dt = \text{Im}\left(\int z(t)dt\right)$$

Therefore the complex integration formula

$$\int e^{(2+3i)t}dt = \frac{1}{2 + 3i}e^{(2+3i)t} + C = \frac{2 - 3i}{(2 + 3i)(2 - 3i)}e^{(2+3i)t} + C$$

$$= \frac{2 - 3i}{2^2 + 3^2}e^{(2+3i)t} + C = \frac{e^{2t}}{13}(2 - 3i)(\cos 3t + i \sin 3t) + C$$

$$= \frac{e^{2t}}{13}[(2 \cos 3t + 3 \sin 3t) + i(2 \sin 3t - 3 \cos 3t)] + C$$

implies that

$$\int e^{2t} \cos 3t \ dt = \frac{e^{2t}}{13}(2 \cos 3t + 3 \sin 3t) + C$$

and

$$\int e^{2t} \sin 3t \ dt = \frac{e^{2t}}{13}(2 \sin 3t - 3 \cos 3t) + C$$