Driven Oscillators: The most important problems involving oscillators have a forcing (or driving) term on the right hand side: We will focus on the problem

\[ mu'' + \gamma u' + ku = F_0 \cos(\omega t), \]

where \( m > 0, \gamma \geq 0, \) and \( k > 0. \)

This problem can be solved by the method of undetermined coefficients. This is quite easy when \( \gamma = 0, \) but it’s a mess when \( \gamma > 0. \) We separate the problem into three cases:

Case I: \( \gamma = 0, \omega \neq \omega_0 = \sqrt{k/m}. \)

Case II: \( \gamma = 0, \omega = \omega_0. \)

Case III: \( \gamma > 0 \)

Examples of Case I and II: Solve the IVPs \( u'' + ku = \cos(3t), u(0) = 0, u'(0) = 0, \) with \( k = 10 \) (Case I) and \( k = 9 \) (Case II).

As shown in class, the solutions are \( u(t) = -\cos(\sqrt{10}t) + \cos(3t), \) and \( u(t) = 1/6t \sin(3t), \) respectively. Here are some graphs of solutions.

Figure 1: The solutions to the IVPs for a short time interval. The solutions look almost identical.

Figure 2: The solutions to the IVPs for a longer time interval.

The envelope of the solution with \( k = 10 \) can be obtained from the trig identities

\[ \cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B). \]
We can combine these together, anticipating that we want an identity for \( u(t) = -\cos(\sqrt{10}t) + \cos(3t) \), where \( \sqrt{10} > 3 \):

\[ -\cos(A + B) + \cos(A - B) = 2\sin(A)\sin(B) \]

Applying this to our problem with \( A + B = \sqrt{10} \) and \( A - B = 3 \), so that \( A = (\sqrt{10} + 3)/2 \approx 3.08 \) and \( B = (\sqrt{10} - 3)/2 \approx 0.0811 \):

\[ u(t) = -\cos(\sqrt{10}t) + \cos(3t) = 2\sin(A t)\sin(B t) = [2\sin(B t)]\sin(A t) \]

The “carrier” wave is \( \sin(A t) \) which has period \( 2\pi/A \approx 2.04 \) and the “envelope” is \( u(t) = \pm 2\sin(B t) \). Now, \( 2\sin(B t) \) has period \( 2\pi/B \approx 77.4 \). The envelope is plotted as a dotted line in Figures 1 and 2. Figure 2 (\( k = 10 \)) shows the phenomenon of “beats,” wherein the amplitude of the oscillation varies with a period of half of 77.4, which is 38.7.

**Case III:** This is the most important case. Any oscillators you will run into will have some damping. The general solution to equation (1) is

\[ u(t) = u_c(t) + U_p(t) \]

where \( u_c(t) \to 0 \) as \( t \to \infty \), since the oscillator is damped (\( \gamma > 0 \)). The *steady state solution*, also called the *forced response*, has the same frequency as the driving term:

\[ U_p(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta) \]

If the \( A, B \) form of the particular solution is substituted into equation (1), we get a system of two linear equations for \( A \) and \( B \) which have the solution

\[ A = \frac{F_0m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad B = \frac{F_0\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \]

Note that these expressions use \( \omega_0 = \sqrt{k/m} \), even though the oscillator is damped. The “rectangular coordinates” \( A \) and \( B \) can be converted to “polar coordinates” \( R \) and \( \delta \), as described in in §3.8. Note that \( B > 0 \), so \( \delta \) is in quadrant I or II. The inverse cosine function gives an angle in quadrant I or II, so the best expressions are

\[ R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \delta = \arccos \left( \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \right) \]

These expressions for the forced response are quite complicated. Nonetheless, we can get a feeling for them by considering limits and drawing figures.

Note that as \( \omega \to 0 \), the response has \( A \to F_0/k \) and \( B \to 0 \). This makes sense, because the right-hand side of (1) is \( F_0 \) when \( \omega = 0 \), and a particular solution in this case is \( U_p(t) = F_0/k \). Furthermore, \( R \to 0 \) as \( \omega \to \infty \).

All of the constants \( m, \gamma \), etc. make the expressions for \( A \) and \( B \) look complicated. However, things are much simpler if we define two dimensionless quantities:

\[ x = \frac{\omega}{\omega_0}, \quad \text{and} \quad Q = \frac{m\omega_0}{\gamma} \]
Here, $x$ is the ratio of the driving frequency ($\omega$) to the natural frequency ($\omega_0$). The amount of damping is measured by the so-called oscillator $Q$, or quality factor. A high $Q$ oscillator has very little damping. With these two variables $x$ and $Q$, the expression for $R$ is more understandable:

$$R = \frac{F_0}{k} \frac{1}{\sqrt{(1-x^2)^2 + (x/Q)^2}}$$

If we treat $Q$ as a constant, then $R$ as a function of $x$ is a maximum at $x_m$, with a maximum value of $R_m$, given by

$$x_m^2 = 1 - \frac{1}{2Q^2}, \quad R_m = \frac{F_0}{k} \frac{Q}{\sqrt{1 - 1/(2Q)^2}} \approx \frac{F_0}{k} \left( Q + \frac{1}{8Q} \right)$$

provided that $Q^2 \geq 1/2$. The approximate expression for $R_m$ is good for $Q$ large. Note that $x_m^2$ is half way between 1 (when $\omega = \omega_0$) and the square of the quasifrequency $\mu$, since it can be shown that $(\mu/\omega_0)^2 = 1 - 1/Q^2$.

Figure 3: The scaled amplitude ($\text{Amp} = Rk/F_0$) of the steady state response as a function of $x = \omega/\omega_0$ for $Q = 10, 5, 2, 1/\sqrt{2}$, and $1/3$. The dotted line goes through the maxima of the curves, $(x_m, R_m)$.

Figure 4: The phase $\delta$ of the steady state response as a function of $x = \omega/\omega_0$ for $Q = 10, 5, 2, 1/\sqrt{2}$, and $1/3$. As $Q \to \infty$, the phase approaches the step function $\delta = 0$ if $x < 1$ and $\delta = \pi$ if $x > 1$. 
Using Complex Numbers: The standard method of calculating of $A$ and $B$ in the particular solution is truly gruesome. However, it is quite easy using (what else?) complex numbers. The ODE (1) has the form

$$L[u(t)] = mu'' + \gamma u' + ku = Re \left( F_0 e^{i\omega t} \right)$$

where $L$ is a linear operator. We look for a solution of the form

$$U_p(t) = A \cos(\omega t) + B \sin(\omega t) = Re \left( (A - i B) e^{i\omega t} \right)$$

When we plug this into the ODE to find $U_p(t)$, we just have to solve

$$L[(A - i B)e^{i\omega t}] = F_0 e^{i\omega t}.$$ 

Example: Find a particular solution to

$$L[u] = u'' + u' + 4u = \cos(\omega t)$$

To find the particular solution, we need to solve

$$L[(A - i B)e^{i\omega t}] = e^{i\omega t}$$

for $A$ and $B$. The operator $L[(A - i B)e^{i\omega t}]$ is easily computed, since $A$ and $B$ are constants. The result is

$$(A - i B)(-\omega^2 + i\omega + 4)e^{i\omega t} = e^{i\omega t}.$$ 

We can divide both sides by $e^{i\omega t}$ to get

$$(A - i B)(-\omega^2 + i\omega + 4) = 1,$$

or

$$(A - i B) = \frac{1}{-\omega^2 + i\omega + 4} = \frac{1}{4 - \omega^2 + i\omega} = \frac{4 - \omega^2 - i\omega}{(4 - \omega^2)^2 + \omega^2}.$$ 

Therefore $A = \frac{4 - \omega^2}{(4 - \omega^2)^2 + \omega^2}$ and $B = \frac{\omega}{(4 - \omega^2)^2 + \omega^2}$. The steady state solution is

$$U_p(t) = \frac{4 - \omega^2}{(4 - \omega^2)^2 + \omega^2} \cos(\omega t) + \frac{\omega}{(4 - \omega^2)^2 + \omega^2} \sin(\omega t)$$

Alternative Method: The way to really do these problems is to let $\hat{A}$ be a complex amplitude and write

$$U_p(t) = Re(\hat{A} e^{i\omega t}) = Re(\hat{A}) \cos(\omega t) - Im(\hat{A}) \sin(\omega t)$$

Then the previous problem becomes:

$$\hat{A}(-\omega^2 + i\omega + 4)e^{i\omega t} = e^{i\omega t}.$$ 

or

$$\hat{A} = \frac{1}{-\omega^2 + i\omega + 4} = \frac{1}{4 - \omega^2 + i\omega} = \frac{4 - \omega^2 - i\omega}{(4 - \omega^2)^2 + \omega^2}.$$ 

This gives the same solution as before. (These two methods are almost the same.)

Extra Credit: Use either of these complex number methods to justify the general formulas (8), (9), and (10) in the book. This is worth 5 class points. Turn it in directly to Prof. Swift by Oct. 15.