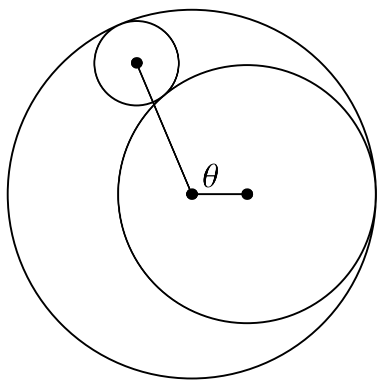
MAT 201 Open Problems, Fall 2011 updated December 1, 2011

3. Analyze the game "Toot and Otto". Does the first player win if he or she uses perfect strategy? Is it always a draw if both players use perfect strategy? Those questions are probably too difficult, so *any* partial analysis of the game is welcome.

Outline of rules for Toot and Otto. Each person has 6 "T" tiles and 6 "O" tiles. The players drop them in a 6 wide by 4 high tray (like the board for connect 4). One person tries to spell TOOT across, down or diagonally. The other tries to spell OTTO.

8. (not open, but needed for open problem 9) A, B, and C are the centers of three circles, each of which is tangent to each of the others. Show that the perimeter of the triangle ABC is equal to the diameter of the largest of the circles.



9. (Hint: Look up circles of Apollonius on the web.) I fudged that picture for problem 8. I started with a circle of radius 1, centered at the origin. That is the big circle. Then I put in a circle of radius 0.7 centered at $(0.3, 0)$. For the third, smallest, circle I found that a circle of radius 0.23 centered at $(-0.3, 0.712)$ looked good. But it is not perfectly tangent to the other 2 circles. Find the possible centers and radii (exactly) of the third circle. Or find one circle that works exactly. If possible, generalize so that the radius 0.7 of the second circle can be any number between 0 and 1.

Restatement of question 9: Let θ be the angle between the centers of the circles, as shown in the updated figure. Find r_3 (the radius of the third circle) as a function of θ and r_2. (For the figure, $r_2 = 0.7$). This effectively answers the original question since the center of the third circle is at $[1 - r_3](\cos\theta, \sin\theta)$

12. Some students asked for a problem that requires differential equations. Here's one that I think can be solved with DEs, but frankly I don't know the solution.

A fox runs at exactly the same speed, v, as his prey, a rabbit. The fox starts at the origin, with coordinates $(0,0)$. The rabbit starts one meter north of the fox, at $(0,1)$. At $t=0$, the rabbit starts running east, so his position is $x=vt, y=1$. The fox also starts running at $t=0$, and always runs straight at the rabbit, with speed v. The frustrated fox will never catch the rabbit.

12a. What curve in the $x-y$ plane will the fox follow? (It starts out with a vertical tangent, like $y=\sqrt{x}$, and has a horizontal asymptote.)

12b. As $t \to \infty$, how far from the rabbit will the fox be?

15. Prove that any quadratic expression $Q(x) = Ax^2 + Bx + C$ can be put uniquely into the form $Q(x) = \frac{k}{2}x(x-1) + \ell x + m$, where k, ℓ, and m depend on the coefficients A, B, and C. Furthermore, prove that $Q(x)$ is an integer for all integers x if and only if k, ℓ, and m are integers.

17. (Requires Calculus III). Evaluate $\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\, dy\, dx$, where a and b are positive.

Hint: If you took Calc 3, you might have done this problem. Evaluate the integral by reversing the order of integration. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

18. A large bucket has 2011 red balls and 2011 white balls, well mixed. Do this until there are fewer than three balls left:

- Pull out three balls at random.
- If the balls are all the same color, throw them all away.
- If there are two of one color and one of the other, throw away the single ball and stir the two of the same color back into the bucket.

What is the probability of each of the 6 possibilities for what is left when you are done?

- No balls
- One red ball
- One white ball
- Two red balls

- Two white balls
- One of each

Note added 10-15: You may use a computer to estimate the probabilities using a Monte Carlo method. But Steve Wilson assures me that this problem can be solved exactly with pencil and paper, at least with 2010 balls.

Note added 10-27: I did NOT reproduce Steve Wilson's problem correctly. This problem might not be solvable with pencil and paper. See problem 30.

19. (not open, but needed for open problem 20) The escalator at the new Health and Learning Center (HLC) will take me one floor up in 15 seconds if I'm standing. If I walk up the moving escalator the time is reduced to 6 seconds. When the escalator is not moving, how long will it take me to walk up the escalator steps. (Ignore relativistic effects.)

20. Redo the escalator problem with the 15 replaced by t_1 and 6 replaced by t_2. Find the last time, t, in terms of t_1 and t_2. At this point the question becomes open ended: Find another choice for t_1 and t_2 that makes t an integer. Find more choices like this. Can you find all choices like this?

23. Find all of the right triangles which have integer length legs (that is, a and b are integers in the famous $a^2 + b^2 = c^2$) *and* the area of the triangle is numerically equal to three times the perimeter of the triangle.

26. Eight distinct points are placed on a circle. Chords are drawn between every pair of points. The points have been placed so that no three chords intersect at a single point. How many triangles are drawn?

Note added October 27: This problem might not be well-posed. See problems 31 and 32.

29. A geometric sequence $\{a_n\}_{n=1}^{\infty}$ has $a_1 = \sin(x)$, $a_2 = \cos(x)$, and $a_3 = \tan(x)$ for some real number x. For what value of n does $a_n = 1 + \cos(x)$?

30. (Modification of problem 18) The setup is like problem 18. However, the second rule is modified:

- Pull out three balls at random.
- (1) If the balls are all the same color, throw them all away.

- (2) **If there are two of one color and one of the other, stir one of the majority color back into the bin, and throw away the other two.**

(The old rule 2 is this: If there are two of one color and one of the other, throw away the single ball and stir the two of the same color back into the bucket.)

One more thing. Steve Wilson's original problem had 2010 balls, and that might make a difference. Try it with both cases: 2010 balls or 2011. Does it matter?

31. (Modification of problem 26.) Eight distinct points are placed on a circle. Chords are drawn between every pair of points. The points have been placed so that no three chords intersect at a single point. How many triangles are drawn **with all three vertices inside the circle**?

32. Is problem 26 well posed? In other words, do you always get the same number of triangles for any placement of the 8 vertices?

33. If you write down every integer from 1 to 1,000,000, how many times do you write the digit 1?

41. The length of the interval of solutions of the inequality $a \leq 2x + 3 \leq b$ is 10. What is $b - a$?

43. Alice drove at an average rate of 80 kph and then stopped for 20 minutes for gas and a snack. After the stop, she drove at an average rate of 100 kph. Altogether she drove 250 km in a total trip time of 3 hours, including the stop. How many hours did she drive before the stop?

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