MAT 136 (Calculus 1), Prof. Jim Swift, NAU
The Key to Computing Limits

Here are two theorems we use all the time, that are not explicitly written in our book.

Theorem 1. *If $f(x) = \tilde{f}(x)$ for all $x \neq c$, then* $\lim_{x\to c} f(x) = \lim_{x\to c} \tilde{f}(x)$

Theorem 2. *If $f(x) = \tilde{f}(x)$ for all $x > c$, then* $\lim_{x\to c^+} f(x) = \lim_{x\to c^+} \tilde{f}(x)$

(There is a similar theorem about the limit of $f(x)$ as x approaches c from the left.) Theorems 1 and 2 are both a consequence of the definition of the limit. The limit does not care a bit about $f(c)$, *unless you know that f is continuous at c.* Recall the definition of continuity:

Definition 1. A function f is *continuous at c* provided $\lim_{x\to c} f(x) = f(c)$.

This means two things:

If $\lim_{x\to c} f(x) = f(c)$, then f is continuous at c.
and
If f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$.

We usually use this second thing, combined with the fact that

Every function we can graph on our calculators,
with the exception of rounding functions,
is continuous on its domain

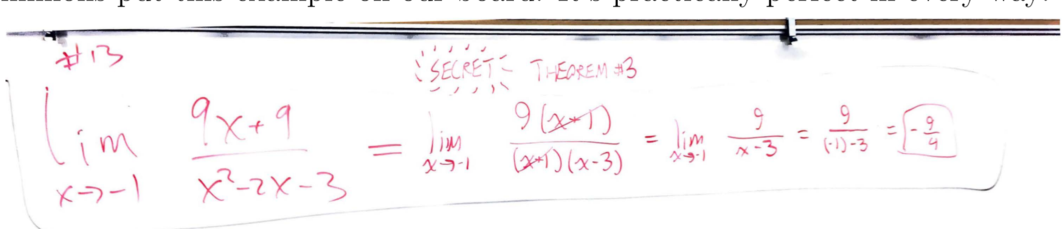
Now combine Theorem 1 and Definition 1 and we get Theorem 3:

Theorem 3. *If $f(x) = \tilde{f}(x)$ for all $x \neq c$, and $\tilde{f}$ is continuous at c, then*

$$\lim_{x\to c} f(x) = \tilde{f}(c).$$

This is how we compute limits by hand. We are usually interested in $\lim_{x\to c} f(x)$ when $f(c)$ is not defined. We find a slightly different function, $\tilde{f}$ that is equal to f for all $x \neq c$, and $\tilde{f}$ is continuous at c.

Kyle Simmons put this example on our board. It's practically perfect in every way:



Take a minute to understand why each of the equal signs is true. Note that you don't need to actually write down "$f(x) = \cdots$" and $\tilde{f}(x) = \cdots$". Just follow this pattern.

As a general rule, always write an = sign if 2 things are equal, and never write an equal sign between two things that aren't equal. Here's a calculation that seems to get the correct answer, but which of the equal signs is true?

$$\lim_{x\to -1} \frac{9x+9}{x^2-2x-3} = \frac{9(x+1)}{(x+1)(x-3)} = \frac{9}{x-3} = \frac{9}{(-1)-3} = -\frac{9}{4}$$

Only the last one! This next line shows all the false equal signs.

$$\lim_{x \to -1} \frac{9x+9}{x^2-2x-3} \neq \frac{9(x+1)}{(x+1)(x-3)} \neq \frac{9}{x-3} \neq \frac{9}{(-1)-3}$$

In an exam, an answer like the one on the previous page would get about one half of the possible points. You can't "borrow from bank and pay it back later." We don't believe "all's well that ends well" in math. We don't think "the ends justify the means." The moral of the story is to follow Kyle's example.

New topic:
Here's another theorem that's not in the book, but we use it all the time.

Theorem 4. *If* $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *both exist, and* $\lim_{x\to c} f(x) \neq 0$*, and* $\lim_{x\to c} g(x) = 0$ *then* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *does not exist.*

What can you conclude from theorem 4 about these limits: (fill in the blanks)

$$\lim_{x\to 0} \frac{1}{x^2} \_\_\_\_\_\_\_\_\_\_ \qquad \lim_{x\to 0} \frac{1}{x} \_\_\_\_\_\_\_\_\_\_$$

Possibly we can be more specific than Theorem 4. If the hypothesis of Theorem 4 holds, it is possible that $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$, or that $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$, or neither.

Note:

$$\left(\lim_{x\to c} \frac{f(x)}{g(x)} = \infty \text{ or } \lim_{x\to c} \frac{f(x)}{g(x)} = -\infty\right) \implies \lim_{x\to c} \frac{f(x)}{g(x)} \text{ DNE.}$$

On the other hand,

$$\lim_{x\to c} \frac{f(x)}{g(x)} \text{ DNE} \quad \not\Longrightarrow \quad \left(\lim_{x\to c} \frac{f(x)}{g(x)} = \infty \text{ or } \lim_{x\to c} \frac{f(x)}{g(x)} = -\infty\right)$$

Sketch the graphs of $y = \frac{1}{x^2}$ (on the left) and $y = \frac{1}{x}$ (on the right) below.

Use these graphs to give the most specific answer possible for these limits:

$$\lim_{x\to 0} \frac{1}{x^2} \_\_\_\_\_\_ \qquad \lim_{x\to 0^+} \frac{1}{x} \_\_\_\_\_\_ \qquad \lim_{x\to 0^-} \frac{1}{x} \_\_\_\_\_\_ \qquad \lim_{x\to 0} \frac{1}{x} \_\_\_\_\_\_$$

Warning: WeBWorK will mark your answer wrong if you write "DNE" when you could write "∞" even though both are correct. They want the most specific answer.