Calculus I Review Sheet
Prof. Swift

There are three key concepts in calculus: The limit, the derivative, and the integral. You need to understand the definitions of these three things, and understand how they relate. You also learned a lot of notation, and applications for the derivative. You also need to be able to “turn the crank” and find the derivative and integral of functions like \( f(x) = x^2 + 3x + 2 \) or \( g(x) = e^x \sin(e^x) \).

The discovery of calculus is one of the crowning achievements of Western civilization, and I hope you have gotten a feeling of how powerful these three concepts (the limit, derivative and integral) are. Please spend the time to go over this review sheet and hopefully get the “big picture” of calculus. It will pay off on the final exam.

The limit: “\( \lim_{x \to a} f(x) = L \)” or “\( f(x) \to L \) as \( x \to 0 \)” means that \( f(x) \) is arbitrarily close to \( L \) for all \( x \) sufficiently close to, but not equal to, \( a \). Thus, the definition of \( \lim_{x \to a} f(x) \) has nothing to do with \( f(a) \). A function \( f \) is continuous at \( a \) iff \( \lim_{x \to a} f(x) = f(a) \). If we know a function is continuous at \( a \) we can use this fact to compute the limit of \( f \) as \( x \) approaches \( a \). Almost all the functions that we’re familiar with are continuous on their domain.

Example: \( f(x) = \frac{x^2 - 1}{x - 1} \) is undefined at \( x = 1 \), so \( f \) is not continuous at 1. We can still consider \( \lim_{x \to 1} f(x) \) since this limit has nothing to do with \( f(1) \). Since

\[
f(x) = \frac{(x + 1)(x - 1)}{x - 1} = x + 1
\]

for all \( x \) except 1, \( \lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2 \). This last limit was evaluated by replacing \( x \) with 1, since \( x + 1 \) is a continuous function, as are all polynomials. Thus, \( \lim_{x \to 1} f(x) = 2 \), even though \( f(1) \) is undefined.

The graph of \( f \) is the line \( y = x + 1 \) with the point (1, 2) removed.

On the other hand, if we do not know that a function is continuous at \( a \), we need to use the definition of continuity to determine if it is continuous.

Example: Let \( f \) be defined by \( f(x) = \sin(\pi/x) \) if \( x \neq 0 \), and \( f(0) = 0 \). Let \( g \) be defined by \( g(x) = x \sin(\pi/x) \) if \( x \neq 0 \) and \( g(0) = 0 \). Determine if \( f \) and \( g \) are continuous at 0. Answer: Since \( f(x) \) takes on values 1, \(-1\) and everything in between for arbitrarily small \( x \neq 0 \), \( \lim_{x \to 0} f(x) \) Does Not Exist (DNE). Hence, \( f \) is not continuous at 0. On the other hand, since \( -|x| \leq x \sin(\pi/x) \leq |x| \) for all \( x \neq 0 \), and \( \lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0 \), the squeeze theorem says that \( \lim_{x \to 0} g(x) = 0 \). Since \( g(0) = 0 \), \( g \) is continuous at 0.

Even though \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist, we can write \( \lim_{x \to 0} \frac{1}{x^2} = \infty \). This means that \( \frac{1}{x^2} \) is arbitrarily large and positive for all \( x \) sufficiently close to, but not equal to, 0.
We also studied one-sided limits. For example, \( \lim_{x \to 0^+} \frac{|x|}{x} = 1 \) and \( \lim_{x \to 0^-} \frac{|x|}{x} = -1 \). An important relationship between the regular limit and the one-sided limits is

\[
\lim_{x \to a} f(x) = L \iff \left( \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \right)
\]

The derivative: \( f'(a) = \lim_{h \to 0} \frac{f(x) - f(a)}{x - a} \) is called the derivative of \( f \) at \( a \). The derivative function is \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \). There are two interpretations of the derivative: (1) \( f'(a) \) is the rate of change of \( f \) with respect to \( x \), at \( x = a \). (2) \( f'(a) \) is the slope of the tangent line to \( y = f(x) \) at \( (a, f(a)) \).

Example: Let \( f(x) = x^2 \). Then \( f'(x) = \lim_{h \to 0} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \).

(The last limit was computed in the first example.)

You are expected to be able to differentiate just about any function using the rules and facts given on the handout entitled “Differentiation Shortcuts.”

Implicit differentiation is just an application of the chain rule. Example: Find \( y' \) for the curve \( x^2 + xy + 2y^2 = 1 \). Take the \( \frac{d}{dx} \) of both sides to get \( 2x + y + xy' + 4yy' = 0 \). Then solve for \( y' = \frac{-2x - y}{x + 4y} \).

An equation of the tangent line to \( y = f(x) \) at \( x = a \) is \( y = f(a) + f'(a)(x - a) \). The tangent line approximation is \( f(x) \approx f(a) + f'(a)(x - a) \) near \( x = a \).

Example: Let \( f(x) = x^2 \). Find an equation of the tangent line to \( y = x^2 \) at \( x = 1 \), and use this to approximate \( 1.1^2 \). \( f(1) = 1 \) and \( f'(x) = 2x \) so \( f'(1) = 2 \). So an equation of the tangent line is \( y = 1 + 2(x - 1) \). The tangent line approximation gives \( 1.1^2 = f(1.1) \approx 1 + 2(0.1) = 1.2 \). This is rather close to the exact square, \( 1.1^2 = 1.21 \).

Related rates problems: The key here is to write an equation connecting the two quantities that holds for all times. Then take the derivative of both sides with respect to time, using the chain rule.

The signs of \( f' \) and \( f'' \) tell about the shape of the graph of \( f \). If \( f'(x) > 0 \) for all \( x \) in an interval \( I \), then \( f \) is increasing on \( I \). However, the converse is not true: \( f(x) = x^3 \) is increasing for all \( x \), even though \( f'(0) = 0 \). Here is a useful theorem that isn’t in the book.

Theorem: Assume that \( f \) is differentiable on the interval \( I \), then

- \( f \) is increasing on \( I \) if and only if \( f'(x) \geq 0 \) for all \( x \) in \( I \) and there is no sub-interval \( J \) contained in \( I \) such that \( f'(x) = 0 \) for all \( x \) in \( J \).

- \( f \) is decreasing on \( I \) if and only if \( f'(x) \leq 0 \) for all \( x \) in \( I \) and there is no sub-interval \( J \) contained in \( I \) such that \( f'(x) = 0 \) for all \( x \) in \( J \).

In particular, \( f \) is increasing if \( f'(x) \geq 0 \) for all \( x \) and \( f'(x) = 0 \) at a finite set of points.
Example: $f(x) = x^3$ is increasing for all $x$, since $f'(x) = 3x^2 \geq 0$ for all $x$ and $f'(x) = 0$ only for a single value of $x$ ($x = 0$).

Another Example: If $g'(x) = x(1-x)$, then $g$ is increasing on the closed interval $[0, 1]$ and $g$ is decreasing on $(-\infty, 0]$ and on $[1, \infty)$.

By definition, $f$ is concave up on an interval $I$ iff $f'$ is increasing on $I$. There is a similar definition for concave down. A point on the graph $y = f(x)$ where $f$ changes concavity is called an inflection point.

The rules for the shape of $y = f(x)$ can be summarized in this table: (Assume that there are no intervals $J$ such that $f'(x) = 0$ or $f''(0) = 0$ for all $x$ in $J$.)

- $f' \geq 0 \iff f$ is increasing
- $f' \leq 0 \iff f$ is decreasing
- $f'' \geq 0 \iff f'$ is increasing $\iff f$ is concave up
- $f'' \leq 0 \iff f'$ is decreasing $\iff f$ is concave down

Example: Consider $f(x) = 2x^3 - 3x^2$. $f'(x) = 6x^2 - 6x = 6x(1-x)$ and $f''(x) = 12x - 6 = 6(2x - 1)$. We can conclude that $f$ is decreasing on $(-\infty, 0]$, increasing on $[0, 1]$, and decreasing on $[1, \infty)$. Furthermore, $f$ is concave down on $(-\infty, 0.5]$ and concave up on $[0.5, \infty)$. The point $(0.5, f(1)) = (0.5, -0.5)$ is an inflection point of $f$.

A function $f$ with domain $D$ as a global minimum at $x = c$ if $f(c) \leq f(x)$ for all $x$ in $D$, in which case $f(c)$ is the global minimum value of $f$. A function $f$ has a local minimum at $x = c$ if $f(c) \leq f(x)$ for all $x$ in some open interval containing $c$. (Note that an endpoint of $D$ cannot be a local minimum.) A critical number of $f$ is a number $c$ in the domain of $f$ such that $f'(c) = 0$ or $f'(c)$ is undefined. An important theorem is: If $f$ has a local minimum or a local maximum at $c$, then $c$ is a critical number of $f$. The converse is not true: If $c$ is a critical number of $f$, then it is not necessarily true that $f$ has a local extremum at $c$.

Example: It is clear from the graphs of $f(x) = x^2$ and $g(x) = |x|$ that both have a local minimum at 0. According to the theorem, 0 must be a critical number of both $f$ and $g$. Indeed it is, because $f'(0) = 0$ and $g'(0)$ is undefined. However, consider $h(x) = x^3$. Since $h'(x) = 3x^2$, 0 is a critical number of $h$, but 0 is neither a local minimum nor a local maximum of $h$.

Therefore, a critical number of $f$ is just a candidate for a local extremum of $f$. The first derivative test for local extrema (p. 281) tells us for sure if the critical number is a local extremum: If $c$ is a critical number of $f$ and $f'(x)$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$. There’s a similar statement for local minima. Furthermore, if $f'$ does not change sign at the critical number $c$, then $c$ is not a local extremum of $f$.

Examples: $f(x) = x^2$ has $f'(x) = 2x$. Since $f'(x)$ changes from negative to positive at $x = 0$, $f$ has a local minimum at $x = 0$. Furthermore, $f$ has
a global minimum at 0 (see p. 308) and $f(0) = 0$ is the global minimum value of $f$.

On the other hand, 0 is a critical number of $h(x) = x^3$, but since $h'(x) = 3x^2$ is positive on both sides of 0, the first derivative test for local extrema says that the the function $h$ does not have a local extremum at 0.

Recall that an inflection point of a function $f$ is a point $(c, f(x))$ where $f$ changes concavity. In this case $c$ is a critical number of $f'$, and furthermore $f'$ has a local extremum at $c$. Also, $f''$ changes sign at $c$.

Examples: $f(x) = x^3 - 3x^2$ has $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6 = 6(x - 1)$. Since $f''(x)$ changes sign at $x = 1$, $f$ has an inflection point at $x = 1$. The inflection point is $(1, f(1)) = (1, -1)$.

On the other hand, $g(x) = (x - 1)^4$ has $g'(x) = 4(x - 1)^3$ and $g''(x) = 6(x - 1)^2$. Even though $g''(1) = 0$, $g''$ does not change sign at $x = 1$, and $g$ has no inflection point.

You should know L'Hôpital's Rule and how to apply it. Example:

$$
\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{(\sin(x))'}{x'} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1
$$

For optimization problems, you need to get a function of one variable whose output you want to maximize or minimize. Then apply calculus techniques to find a global maximum or minimum value of the function. The first derivative test for global extreme values (p. 308) is helpful.

Newton's method is a way to solve $f(x) = 0$. Starting with an initial guess $x_1$, get a sequence $x_2, x_3, x_4, \text{etc.}$ by the rule $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. This works wonderfully on a graphing calculator.

Example: To solve $f(x) = x^3 - 2 = 0$, starting with $x_1 = 2$, do this on

your calculator: 2 $\rightarrow$ X Enter, $X - (X^3 - 2)/(3X^2) \rightarrow$ X Enter. Now $X = x_2 = 1.5$. Continue hitting Enter until the number stays the same.

The solution is $x = 1.2599$....

The Integral: $\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$ is the definite integral of $f$ from $a$ to $b$. Here, $\Delta x = (b - a)/n$, $x_0 = a$, and $x_i = a + i\Delta x$ so $x_n = b$. We call $f(x)$ the integrand. If $a < b$ this number is equal to the area under the graph of $f$ and above the $x$-axis, minus the area above the graph of $f$ and below the $x$-axis. The Riemann sum using right endpoints is $R_n = \sum_{i=1}^{n} f(x_i) \Delta x$, so $\int_a^b f(x) \, dx = \lim_{n \to \infty} R_n$ by definition. We can also compute Riemann sums using left endpoints ($L_n$), or midpoints ($M_n$). These Riemann sums can be used to approximate a definite integral. MAT 137 (Calculus 2) will have more about applications of the integral.
The Connection between the Derivative and the Integral: If we can find an antiderivative of the integrand of a definite integral we can use the evaluation theorem (also called the Fundamental Theorem of Calculus 2, or FTC2) to compute the integral: \( \int_a^b f(x) \, dx = F(b) - F(a) \), where \( F' = f \). (In other words, \( F \) is an antiderivative of \( f \).) The most general antiderivative of \( f \) is written as the indefinite integral: \( \int f(x) \, dx = F(x) + C \).

Example: \( \int 2x \, dx = x^2 + C \) is the most general antiderivative of \( f(x) = 2x \). We can use this to evaluate \( \int_3^0 2x \, dx = x^2 \Big|_3^0 = 9 - 0^2 = 9 \).

Does every continuous function have an antiderivative? Yes. The FTC1 says that the antiderivative of \( f(x) \) that satisfies \( F(a) = 0 \) is \( F(x) = \int_a^x f(t) \, dt \). For example the antiderivative of \( f(x) = x^2 \) with \( F(1) = 0 \) is

\[
F(x) = \int_1^x f(t) \, dt = \int_1^x t^2 \, dt = \frac{t^3}{3} \Big|_1^x = \frac{x^3}{3} - \frac{1}{3}.
\]

It is easy to verify that \( F'(x) = x^2 \) and \( F(1) = 0 \).

We can write the two fundamental theorems of calculus without mixing \( f \) and \( F \):

\[
\text{FTC1: } \int_a^b F'(x) \, dx = F(b) - F(a) \quad \text{FTC2: } \frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

An elementary function is one that involves sums, products, inverses and compositions of power functions, trigonometric functions, and exponentials. Unfortunately, many elementary functions do not have antiderivatives which are elementary functions. Examples of such functions are \( e^{-x^2} \), \( \sin\left(\frac{x}{x}\right) \), \( \cos(x^2) \), and \( \sqrt{1 + x^3} \). We say that \( \int e^{-x^2} \, dx \) is not an elementary integral. The simplest expression for the antiderivative of \( f(x) = e^{-x^2} \) that satisfies \( F(0) = 0 \) is \( F(x) = \int_0^x e^{-t^2} \, dt \), and \( F(x) \) is not an elementary function.

We have rules that allow us to differentiate just about any function. On the contrary, we do not have enough rules to allow us to integrate any function. So how do we “do” an integral? There are three techniques we learned in this class: (1) recognize, (2) simplify, and (3) substitute. (You will learn more techniques in Calculus II.)

First of all, try to recognize the integrand as the result of differentiating some function. For example, \( \int \sec^2(x) \, dx = \tan(x) + C \), since we know that \( \frac{d}{dx} \tan(x) = \sec^2(x) \). The integrals you should recognize are numbers 2-8, 16 and 17 on reference page 6 at the back of the book.

The second technique is to use algebraic simplification to write the integrand as a sum or difference of terms that we can integrate. Here are some examples:

\[
\int x(2 - 3x) \, dx = \int (2x - 3x^2) \, dx = x^2 - x^3 + C
\]
\[ \int \frac{x^2 + 1}{x} \, dx = \int \left( x + \frac{1}{x} \right) \, dx = \frac{x^2}{2} + \ln |x| + C \]

**Warning:** The integral of a product is not the product of integrals:

\[ \int x(2 - 3x) \, dx \neq \frac{x^2}{2}(2x - \frac{3x^2}{2}) + C. \]

We learned one more technique of integration: \( u \)-substitution.

**Example of \( u \)-substitution:** To integrate \( \int 2x \cos(x^2) \, dx \), let \( u = x^2 \), so \( du = 2xdx \). Then \( \int 2x \cos(x^2) \, dx = \int \cos(u) \, du = \sin(u) + C = \sin(x^2) + C. \)

**Warning:** While it is true that \( \int u^2 \, du = \frac{u^3}{3} + C, \)

\[ \int \sin^2 x \, dx \neq \frac{\sin^3 x}{3} + C \]

If \( u = \sin(x) \), then \( du = -\cos(x) \, dx \) so a proper use of this \( u \) substitution is:

\[ \int \sin^2 x \cos x \, dx = -\int u^2 \, du = -\frac{u^3}{3} + C = -\frac{\sin^3 x}{3} + C \]

To do a \( u \)-substitution in a definite integral there are two methods: Method 1 is to do the substitution first for the indefinite integral. For example, using the example of \( u \) substitution above we found that \( \int 2x \cos(x^2) \, dx = \sin(x^2) + C \). Therefore

\[ \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx = \sin(x^2)\bigg|_0^{\sqrt{\pi}} = \sin(\sqrt{\pi}^2) - \sin(0) = \sin(\pi) - \sin(0) = 0. \]

Method 2 is to re-write the limits in terms of \( u \). For example with \( u = x^2 \), the lower limit \( x = 0 \) corresponds to the lower limit \( u = 0^2 = 0 \), and the upper limit \( x = \sqrt{\pi} \) corresponds to the upper limit \( u = \sqrt{\pi}^2 = \pi \). Therefore

\[ \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx = \int_0^{\pi} \cos(u) \, du = \sin(u)\bigg|_0^{\pi} = \sin(\pi) - \sin(0) = 0. \]

**Warning:** Note that

\[ \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx \neq \int_0^{\sqrt{\pi}} \cos(u) \, du, \]

since \( \int_0^{\sqrt{\pi}} 2x \cos(x^2) \, dx = 0 \) while

\[ \int_0^{\sqrt{\pi}} \cos(u) \, du = \sin(u)\bigg|_0^{\sqrt{\pi}} = \sin(\sqrt{\pi}) - \sin(0) = \sin(\sqrt{\pi}) \approx 0.98. \]