Euler's treatises on infinitesimal analysis: *Introductio in analysin infinitorum, Institutiones calculi differentialis*, *Institutionum calculi integralis*.  

Giovanni Ferraro

**Introduction**

During the eighteenth century mathematics underwent a period of significant development which did not just consist of the mere addition of new results, techniques and procedures to old ones but of a profound change in its structure, its basic principles and in its underlying epistemological approach. Euler was the main protagonist of this evolutionary process, to which he contributed by means of an enormous quantity of papers that include hundreds of articles and many treatises. In this paper I will describe the main aspects of three of these treatises: *Introductio in analysin infinitorum, Institutiones calculi differentialis*, and *Institutionum calculi integralis*, where Euler offered a systematic exposition of analysis of infinite quantities. They were not didactic works in the sense that they did not aim to introduce and explain a consolidated scientific discipline to students. Instead, Euler's main goal in writing them was the constitution of analysis as a new and autonomous branch of mathematics which was independent of geometry and arithmetic.

In section 1, I will illustrate Euler's concepts of analysis and deal with various selected topics from *Introductio*. I will also discuss Euler's treatment of analytical geometry. In section 2, I will examine Euler's foundation of the calculus and, in particular, the concept of differentials as fictitious quantities and his attempt to consider differential calculus as a calculus of finite quantities, whose objects are the differential coefficients $dy/dx$. I will also describe some applications of differential calculus to series theory. In section 3, I will investigate the problematic concept of integration as an anti-differential and illustrate some of the results contained in the three weighty volumes of *Institutionum calculi integralis*. Finally, I will examine Euler's treatment of functions of more than one variable.
§1 Introductio in analysin infinitorum

The Introductio in analysin infinitorum, published in 1748, was already completed in 1745 (see [EI]). It is composed of two books, which are numbered as E101 and E102 in the Eneström Index. The first book contains an Epistola dedicatoria by Bousquet, the publisher, a Praefatio and eighteen chapters, where Euler deals with elementary functions, infinite series, infinite products, and continued fractions without using differential and integral calculus. In the second part of the Introductio in analysin infinitorum, Euler applies various analytical notions, which he had introduced in the first part, to the study of geometry. The second book contains twenty-two chapters, devoted to analytical geometry in the plane, and an appendix of six chapters on surfaces in space.

Introductio in analysin infinitorum auctore Leonharde Eulero professore regio berolinensis et academia imperialis scientiarum Petropolitanae socio
Epistola dedicatoria
Praefatio

Liber primis continens explicationem de functionibus quantitatum variabilium, earum resolutione in factores, atque evolucione per series infinitas: una cum doctrina de logarithmis, arcibus circularibus, eorumque sinus et tangentiibus; pluribusque aliis rebus, quibus analysis infinitorum non mediocriter adjuvatur.

1. De functionibus in genere.
2. De transformatione functionum.
3. De transformatione functionum per substitutionem.
4. De explicatione functionum per series infinitas.
5. De functionibus duarum pluriumve variabilium.
6. De quantitatibus exponentialibus ac logarithmis.
7. De quantitaturn exponentialium ac logarithmorum per series explicatione.
8. De quantitatibus transcendentibus ex circulo ortis.
10. De usu factorum inventorum in definendi summis seriern infinitorum.
11. De alis arcuum atque sinus expressionibus infinitis.
12. De reali functionum fractarum evolucione.
14. De multiplicatione ac divisione angulorum.

Liber secundi continens theoriaem lineae curvarum

1. De linea curvis in genere.
2. De coordinatarum permutatione.
3. De linearum curvarum algebraicarum in ordinem divisione.
4. De linearum cuiusque ordinis praeципuis proprietatibus.
§1.1 The role of analysis in Euler’s mathematics

When Euler began his mathematical career, in the late 1720s, analysis was still a method for solving geometrical problems. It consisted in representing geometrical quantities (such as ordinate, abscissa, arc length, subtangent, normal, areas between curves and axes) by means of symbols and appropriately manipulating them. Analysis served to improve the knowledge of geometrical entities. It did not have its own foundation: it was only a method of geometry.

A new conception of analysis developed during the 1730s and 1740s. One of the more significant moments in this evolution was certainly the publication of Euler’s treatise on kinematics and dynamics of a point-mass, Mechanica sive motus scientia analytice exposita [1736a], where Euler for the first time applied analytic methods to mechanics in a systematic way. He explained the reasons of his approach as follows:
What pertains to all the works composed without analysis, is particularly true for mechanics. In fact, the reader, even though he is persuaded about the truth of the things that are demonstrated, nonetheless cannot understand them clearly and distinctly. So he is hardly able to solve with his own strengths the same problems, when they are changed just a little, if he does not inspect them with the help of analysis and if he does not develop the propositions into the analytical methods. This is exactly what happened to me, when I began to study in detail Newton’s *Principia* and Hermann’s *Phoronomia*. In fact, even though I thought that I could understand the solution to numerous problems well enough, I could not solve problems that were slightly different. Therefore I strove, as much as I could, to get at the analysis behind those synthetic methods in order, for my purposes, to deal with those propositions in terms of analysis. Thanks to this procedure I perceived a remarkable improvement of my knowledge. (Euler [1736a, 1: 8]. Translation in Guicciardini [1999, 247]).

The *Mechanica* originated from Leibniz’s program to reformulate Newton’s *Principia* in terms of the Leibnizian calculus\(^1\) and can be considered “as a work of systematization of results achieved mainly in the Bernoullian school” (Guicciardini [1999, 248]). However Euler went beyond the intention of previous mathematicians, none of them had showed awareness of the possibility of transforming analytical methods into a new and autonomous field of mathematics. According to Fraser: “Although the theme of analysis was well established at that time there was in his work something new, the beginning of an explicit awareness of the distinction between analytical and geometrical methods and an emphasis on the desirability of the former in proving theorems of the calculus” (Fraser [1997, 63]). In the two books of the *Mechanica* analytical methods were conceived as the peculiar methodology of a general and abstract discipline (analysis), which, just because it was abstract and general, could be applied to a more specific and concrete discipline (mechanics).

This leads us to the crucial question of the role of analysis in Euler’s mathematics and its relationship with geometry and mechanics. Euler thought that analysis investigated the relations between variable quantities in all

\(^1\)On this topic, see Guicciardini [1999].
possible generality: it was the science of general quantity. In his opinion, a quantity\(^2\) was general insofar as one made the values and other specific characters abstract. Therefore, general quantities were abstract entities which were merely characterized by the fact that they could be increased or decreased indefinitely. They were represented by means of symbols of the type \(x, y, \ldots\) and the investigation of their relationships was reduced to the modality of the combinations of these symbols.

While analysis investigated quantity in general, abstract and symbolic manners, geometry, mechanics and arithmetic investigated special types of quantities. In his Vollständige Anleitung zur Algebra, Euler stated that arithmetic investigated numbers in the proper sense of the terms (natural and rational numbers) and dealt with the common way of calculating with numbers, namely it investigated the operations between specific numbers which were not represented by means of the algebraic symbols \(x, y, \ldots\) (cf. Euler [1770, 11]).

It is to be emphasized the notion of a set of real numbers was lacking in Euler. Quantity could assume any numerical value, however it could not be reduced to a set of numbers. According to Euler, a number was the relation of a quantity to another quantity, taken arbitrarily as a unity [1770, 10]. Quantity was therefore considered as an entity that logically precedes number and number was viewed as a tool for treating quantity. In effect Euler termed number any symbolic entity which can be manipulated in a similar way to natural numbers (on Euler’s concept of number, see Ferraro [2004]).

Euler conceived geometry as the science of geometric quantities. Geometric quantities were abstract entities too. However, unlike general quantities, they were the idealization of empirical objects and were given by diagrams. In other terms, a geometric quantity was given by means of iconic representations which imitated the essential character of the empirical object from which the geometric quantity was drawn. The difference between geometric and general quantities involved different methods for dealing with them. When one examined geometric quantities, one used diagrammatic representations and diagrams were an essential part of the reasoning. Geometry was entrusted, to a certain extent, to the intuitive immediacy of an inspection of a diagram and the perception of the relationships shown in the diagram. For instance, the geometric intuition connected to a geometric diagram ensured

\(^2\)On Euler’s notion of quantity, see Ferraro [1998] and [2003]
that the triangle $T$ and the segment $s$ had a common point (Figure 1).

\[ \begin{array}{c}
\text{Figure 1: Relationships shown in a diagram.}
\end{array} \]

Instead, analysis functioned in a discursive way along abstract notions; it was based upon a linguistic deduction and did not unload a part of the reasoning on to diagrams (however, this does not mean that deduction derived from explicit axioms). In Euler’s opinion, analytical symbols were the instruments which allowed analysis to be an abstract, conceptual, and merely discursive theory, which did not rely on material representations, whereas geometry was concrete (or more concrete) theory and relied on the aid of diagrams. Thus, when Euler claimed the absence of geometric figures in his analytic treatises\(^3\), he asserted the absence of inference derived from the mere inspection of a figure which was crucial in classical geometric proofs (on the use of diagrams in Greek geometry, see Netz [1999]).

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In analysis symbolic expressions were manipulated according to given rules. These rules were the immediate generalizations of the laws that rule arithmetic operations or the extension of certain properties of geometrical quantity. Thus even though it is true that Euler’s analysis actually dispensed with diagrams, the analytical procedures retained an implicit geometric characterization. Indeed a function was thought to possess the usual properties of a “nice” curve, lack of jumps, presence of the tangent, curvature radius, etc. In modern terms, functions were thought to have properties we can call continuity, differentiability, Taylor expansion, etc.... For instance, in Eu-

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\(^3\)In the preface to *Institutiones calculi differentialis*, Euler stated: “I mention nothing of the use of this calculus in the geometry of curved lines, because its absence will be least felt, since it has been investigated so comprehensively that even the first principles of differential calculus are, so to speak, derived from geometry and, as soon as they had been sufficiently developed, were applied with extreme care to this science. Here, instead, everything is contained within the limits of pure analysis so that no figure is necessary to explain the rules of this calculus” [1755, 9].
ler’s opinion it was obvious that (save exceptional points) each function \( y(x) \) possessed the following property:

\[
(P) \quad \Delta y = y(x + \omega) - y(x) \text{ is infinitesimal if } \omega \text{ is infinitesimal} \quad \text{(cf. Euler [1755, 1: §.113]).}
\]

This property expressed geometric continuity of curved lines in analytical terms.

In Euler’s view, analysis was the most general (and the only really pure) part of mathematics, whereas arithmetic, geometry and mechanics tend to be regarded as fields of application of pure analysis. In the *Introductio* Euler made this conception explicit; thus, in the first book he treated analysis of general quantities and then, in the second book, he applied analytical notions to geometry.

It should also be noted that Euler divided analysis into three parts: the analysis of finite quantities, the introduction of the analysis of infinite quantities and the calculus. Indeed, in the *Introductio*, Euler collected the part of the analysis of infinite quantities where the operations of differentiation and integration were not used. The first book of the *Introductio* constituted a corpus of knowledge which was midway between the analysis of finite quantities and the calculus (the higher analysis), where the notion of the differential and the principle of cancellation of higher-order differentials⁴ were used. This distinction was largely shared during the eighteenth century and the topics of the *Introductio* became a specific field of analysis, later termed as algebraic analysis (see Lacroix [1797]).

Euler’s tripartite division of analysis was also the manifestation of his aim to reduce analysis as far as possible to algebraic notions; this latter term is used here to refer to notions deriving from an infinitary extension of the principles of analysis of finite quantities. Euler was not entirely successful in achieving his aim, since he introduced infinitesimal considerations in various proofs; however, algebraic analysis, as a particular field of mathematics, was clearly set out in the *Introductio*. At the end of the eighteenth century, Euler’s plan to undertake an algebraic treatment of the broadest possible part of analysis of infinity had far-reaching consequences when Lagrange tried to reduce the whole of calculus to algebraic notions (see Ferraro-Panza [L]).

⁴The principle of cancellation stated that \( dx^n + dx^m \) is equal to \( dx^n \) if \( n < m \).
§1.2 Selected topics from the first book of the Introductio

The heart of Euler’s analysis was the concept of a function. In Euler’s conception a function was a pair which consisted of a quantity\(^5\) and the analytical expression of this quantity\(^6\). In the Introductio Euler investigated in detail elementary functions, namely algebraic, exponential, logarithmic, and trigonometric functions (on the use of functions different from the elementary ones in Euler’s mathematics, see section 3.1). Euler did not really define transcendental elementary functions if the term “definition” is taken to mean a free act of will by which we create the definitendum\(^7\). In the case of the exponential function, Euler initially observed that \(a^z\) had a meaning when \(z\) was a natural number. Then he considered the case in which \(z\) was a negative integer or zero. He, later, observed that, if \(z\) was a fraction, such as \(z = 5/2\), the quantity \(a^z\) assumed a unique positive real value \((a^{2\sqrt{2}})\), which lay between \(a^2\) and \(a^3\). A similar situation occurred if \(z\) was irrational: for example, the quantity \(a^{\sqrt{2}}\) had a determined value lying between \(a^2\) and \(a^3\) (Euler [1748, 1: §97]). At first sight, it would seem that Euler defined the exponential function \(a^z\) by associating a real value with the symbol \(y = a^z\) for each real number \(z\). In reality Euler, who had no theory of real numbers, sought analytically to characterize a quantity \(y\), represented by the symbol \(a^z\), by assuming the existence of this quantity. The construction of \(a^z\) was an example of Wallis’s interpolation\(^8\), i.e., it was the solution to the problem:

Find a quantity \(y = a^z\) that interpolates ..., \(a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots\)

Euler did not give any definition of the sine and cosine: he considered ‘\(\sin x\)’ and ‘\(\cos x\)’ as symbols satisfying properties such as \(\sin(x + y) = \sin x \cos y + \cos x \sin y\) and \(\sin^2 x + \cos^2 x = 1\) [1748, 1: §§127–130].

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\(^5\)When the term ‘quantity’ is used in connection to the concept of a function it is to be understood in the sense of “quantity depending on other quantities” or “relation between a quantity with other quantity”.

\(^6\)I refer to Panza’s paper in this book for a general investigation of the concept of a function in the Introductio. See also Fraser [1989] and Ferraro [2000]. In the last paper I also discussed the definition of a function that Euler gave in his [1755] and showed that there is no substantial difference between Euler’s concept of a function in [1748] and [1755].

\(^7\)Only in the case of the logarithm, Euler gave an acceptable definition \((z = \log y\) if \(y = a^z\)) [1748, §102].

\(^8\)On Euler’s interpretation of Wallis’s interpolation, see Ferraro [1998].
Power series were the main instrument Euler used for investigating functions. They were not themselves regarded as functions but were viewed as the expansion of given functions and served to improve the understanding of these functions (see Fraser [1989]). In the *Introductio* Euler mainly used two rules to expand functions into series:

a) the method of indeterminate coefficients.

b) the binomial theorem.

These two rules were based upon the principle of infinite extension of rules and procedures, according to which the rules that were valid for finite expressions could be applied to infinite expressions and in an unending number of steps.

An example of the method of indeterminate coefficients is the following. In order to expand the function \( \frac{c}{b+x} \), one posed \( \frac{c}{b+x} = A + Bx + Cx^2 + Dx^3 + \cdots \) and determined the coefficients \( A, B, C, \ldots \), as if \( A + Bx + Cx^2 + Dx^3 + \cdots \) was a finite polynomial. Indeed

\[
c = (b + x)(A + Bx + Cx^2 + Dx^3 + \cdots)
= Ab + Bbx + Cbx^2 + Dbx^3 + \cdots
\]

\[
= Ax + Bx^2 + Cx^3 + \cdots
= \frac{c}{b + x}
\]

Since \( c = Ab + (Bb + A)x + (Cb + B)x^2 + (Db + C)x^3 + \cdots \), one obtained an infinite system of equations

\[
c = Ab, Bb + A = 0, Cb + B = 0, Db + C = 0, \ldots.
\]

By solving this system, one determined the coefficients \( A, B, C, \ldots \) and had the sought expansion

\[
\frac{c}{b + x} = \frac{c}{b} - \frac{c}{b^2} x + \frac{c}{b^3} x^2 - \cdots.
\]

The binomial expansion is the relation

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \cdots.
\]

Newton had obtained it merely by considering the expansion of \((1 + x)^n\) for an integer \( n \) and extending the formula to rational numbers \( n \). In the *Introductio* Euler did not provide any proof of the binomial expansion. In
the years that followed Euler attempted to prove it in his [1755], [1774–75]
and [1787].

Euler used (1.1) to expand several functions. For instance, in the case
of the exponential function, he considered the equality \(a^\omega = 1 + \psi\), where
\(\omega\) and \(\psi\) are infinitesimal, and assumed that \(\psi\) was equal to \(k\omega\) and that
\(a\omega = 1 + k\omega\). Then he put \(i = x/\omega\), where \(x\) is a finite number, and observed that
\[
a^x = a^{i\omega} = (1 + k\omega)^i = \sum_{r=0}^{\infty} \binom{i}{r} (k\omega)^r = \sum_{r=0}^{\infty} \binom{i}{r} \left(\frac{kx}{i}\right)^r.
\]

Euler asserted that \(\frac{i-1}{i} = 1, \frac{i-2}{i} = 1, \frac{i-3}{i} = 1, \ldots\) for an infinitely large
number\(^9\) \(i\). Thus he obtained
\[
(1.2) \quad a^x = \sum_{r=0}^{\infty} \frac{1}{r!} (kx)^r.
\]

(Euler [1748, 1:122–124]).

By putting \((1 + k\omega)^i = 1 + x\), Euler obtained \(i\omega = \frac{1}{k}(1 + x)^{1/i} - \frac{1}{k}\) and
\[
\log_a(1 + x) = \frac{i}{k} \left( (1 + x)^{1/i} - 1 \right).
\]

For \(k = 1\), he had \(\log(1 + x) = i((1 + x)^{1/i} - 1)\). By applying (1.1) he derived
\[
(1.3) \quad \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad [1748, 1:125–126].
\]

In chapter 8 Euler showed that \((\cos z + \sqrt{-1}\sin z)^n = \cos nz + \sqrt{-1}\sin nz\).
Hence
\[
\cos nz = \frac{(\cos z + \sqrt{-1}\sin z)^n + (\cos z + \sqrt{-1}\sin z)^{-n}}{2}.
\]

Another application of the binomial theorem allowed him to obtain
\[
\cos nz = \cos^n z - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} z \sin^2 z + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^{n-4} z \sin^4 z + \cdots
\]

Euler put \(nz = v\), where \(z\) was an infinitesimal, \(n\) an infinitely large number
and \(v\) a finite number. In this case \(\sin z = z\) and \(\cos z = 1\), hence
\[
\cos v = 1 - \frac{1}{1 \cdot 2} v^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} v^4 + \cdots \quad [1748, 1: \S\S. 132–133].
\]

\(^9\)On Euler’s use of infinitesimal and infinite numbers, see section 3.1.
In the *Introductio* Euler devoted several pages to recurrent series. A series is said to be recurrent if its general term $a_n$ is a linear combination of a fixed number of antecedent terms, i.e., $a_n = b_1a_{n-1} + b_2a_{n-2} + \cdots + b_n a_{n-s}$; the constants $(b_1, \ldots, b_n)$ are called the scale of series. Following de Moivre's and Bernoulli's works and using an easy application of the method of the indeterminate coefficients, Euler proved that the expansion of the function

$$
\frac{c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}}{1 + b_1x + b_2x^2 + \cdots + b_nx^n}
$$

is the series $\sum_{n=0}^{\infty} a_n x^n$ with the scale of recurrence $(-b_n, \ldots, b_2, -b_1)$ and the first $n$ terms $a_0, a_1, a_2, \ldots, a_{n-1}$ are defined by the relations

\begin{align*}
a_0 &= c_0 \\
a_1 &= c_1 - b_1a_0 \\
a_2 &= c_2 - b_2a_0 - b_1a_1 \\
&\vdots \\
a_{n-1} &= c_{n-1} + b_{n-1}a_0 + b_{n-2}a_1 + \cdots + b_1a_{n-2} \text{ (see [1748, 1:§. 212]).}
\end{align*}

Euler also explained how to invert the above procedure. For example, given the recurrent series $A + Bx + Cx^2 + Dx^3 + \cdots$ with the scale $(\alpha, -\beta, \gamma, -\delta)$, he showed that

$$
A + Bx + Cx^2 + Dx^3 + \cdots = \frac{a + bx + cx^2 + dx^3}{1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4},
$$

where $a = A$, $b = B - \alpha A$, $c = C - \alpha B + \beta$, $d = D - \alpha C + \beta B - \gamma A$ [1748, 1: §.231]. On this occasion, he expressly considered the generating function as the sum of the series:

It is clear that the sum of recurrent series is equal to the fraction which generates it [1748, 1: §. 231].

This notion was the basis for the famous Euler’s definition of the sum of a series, which was later formulated as follows:

I term the sum of an infinite series to be the finite expression, from the expansion of which the series is generated [1755, 1: §.111].

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10 See also Euler [1754-55]. I refer to Ferraro [ST] for a discussion of this definition.
In other words, given a function series $\sum f_n(x)$, according to Euler, the sum $f(x)$ of $\sum f_n(x)$ is a function $f(x)$ such that its development is equal to $\sum f_n(x)$.

§1.3 Geometry in the *Introductio in analysin infinitorum*

While Euler considered analysis and geometry to be separate branches of mathematics, he did not think that the application of analysis to geometry involved creating a new mathematical discipline. In his view, geometry meant the investigation of geometrical objects such as lines and planes. The existence of the basic objects of geometry did not rely on implicit or explicit definitions: they were not described verbally but were intuitively given. Lines or planes could only be conceived by means of their representation in a diagram. (It is to be specified that Euler did not necessarily seem to think of a diagram on the sheet: a line could be drawn in thought.)

The investigation of geometrical objects could be performed in two ways:

a) the sole use of appropriate diagrams and verbal reasonings about diagrams (the synthetic method);

b) the additional use of analytical symbols and the results of analysis as well as figurative representations (the analytical method).

It is clear that Euler was fond of analytical methods and thought that the analytical investigation of curves could provide more results than synthetic investigation$^{11}$, especially in the case of more complicated curves. The aim of the *Introductio* was also to provide a proof of the power and efficacy of the application of the analytical method to geometry. However, the basic objects of geometry were only given synthetically; for this reason, the analytical method could not entirely replace the synthetic method in geometry and the idea that analytical geometry could be a branch of geometry that was separate from synthetic geometry did not appear in Euler. In Euler's view, there is no actual distinction between analytic geometry and synthetic geometry. Geometry was a sole entity in terms of subject-matter (curved lines, surfaces, etc.): according to the circumstances, geometric objects could be investigated using the analytical method or the synthetic method.

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$^{11}$In [1748, 2: §. 8] Euler stated that although some curves could be described mechanically, he aimed to study curves insofar as they were originated by functions because this method was the most general and best suited to the calculus.
A consequence of this view is that the analytical study of straight lines (and of circles) was of little importance in the *Introductio*. According to Euler, the geometry of the straight line was well-known and therefore it was not necessary to dwell upon it (cf. [1748, 2: §. 85]). Indeed he limited himself to a brief consideration of the general equation $ax + by = c$ of the straight line. In a similar way, in the appendix, when Euler investigated surfaces, his treatment of the equation $ax + by + cz = d$ of the plane was very brief. Moreover, he did not consider questions concerning the determination of angles and distances. For instance, he did not provide a formula for the distance of two points; if necessary, he calculated the distance of two points by applying the Pythagorean theorem.

It was only at the end of the eighteenth century that the most elementary geometrical concepts (distances, directions, etc.) were expressed in analytic language. In this way analytic formulas, quite independent of reference to diagrams, replaced synthetic theorems and the calculations were carried out with full generality (cf. Boyer [1959, 204–206]). This was the very beginning of analytical geometry as a new branch of geometry autonomous from synthetic geometry\textsuperscript{12}.

In the process that led to the rise of analytic geometry in a modern sense, the *Introductio* was, however, significant for several reasons: 1) the new role of analysis and geometry in mathematics, 2) the use of the notions of a variable and function as the basis for a more general treatment of curved lines, 3) the analytic treatment of spatial surfaces.

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In the first chapter of the second part of the *Introductio*, Euler briefly illustrated a general theory of curved lines. In particular, he introduced a distinction between continuous and discontinuous curves, which was based upon a concept of continuity that differed considerably from the modern one.

\textsuperscript{12} It is worthwhile noting that even if the phrase ‘analytic geometry’ was already used in Newton’s *Geometria analytica sive specimina artis analyticae*, which was published in *Isaac Newtoni opera quae exstant omnia. Commentariis illustrabat Samuel Horsley* (1779), the use of the expression ‘analytic geometry’ to denote a new branch of geometry was proposed by Lacroix in 1797. In the introduction to his *Traité du calcul différentiel et du calcul intégral* [1797] he stated: “There exists a way of viewing geometry that could be called analytic geometry, and which would consist in deducing the properties of extension from the least possible number of principles, and by purely analytic methods”.
Euler viewed continuity as a global matter: it was equivalent to uniqueness. This conception was grounded in the idea that an object was continuous if it was an unbroken object, i.e., if it was not broken in two objects and was therefore one object. (I shall call Euler’s concept of continuity G-continuity for short).

In the *Introductio*, Euler applied this idea of continuity to curves. He stated:

A continuous curve is one such that its nature can be expressed by a single function of $x$. If a curve is of such a nature that different functions of $x$ are required for the expression of its various parts BM, MD, DM, etc., then we call such a curve discontinuous or mixed or irregular. This is because such a curve cannot be expressed by one constant law, but is formed from several continuous parts (Euler [1748, 2: §.9])

For instance, the hyperbola of equation $y = k/x$ is broken into two pieces, however Euler considered it as a continuous curve since it is determined by only one function $y = k/x$. The number of the branches of a curve was therefore of no importance. Instead the curve

$$y = \begin{cases} 
  x & \text{if } x \text{ is a positive quantity} \\
  x^2 & \text{if } x \text{ is a nonpositive quantity}
\end{cases}$$

was considered to be a discontinuous curve since it was not expressed by one only analytical expression.

Euler used a similar criterion to subdivide curves into complex and non-complex ones. He noted that the equation of certain algebraic curves could be broken down in rational factors:

Such equations include not one but many continuous curves, each of which can be expressed by a peculiar equation. They are connected with each other only because their equations are multiplied mutually. Since their link depends upon our discretion, such curved lines cannot be classified as constituting a single continuous line. Such equations (referred to above as complex) do not give rise to continuous curves, although we shall call for instance since $y^2 = x$.

In the $h$ dental and equations. He analytic was second degree:

$$y = \frac{k}{x}$$

and deduced difference between derived from “what means” Euler’s derived. In “what means” (1748) ch 2x (1.4) by using an that.

1) if $\gamma > 0$ the curve be hyperbola.

2) if $\gamma < 0$ Therefore the is an ellipse.

3) if $\gamma = 0$ according to, $y = \pm \infty$. If the curve be an ellipse.

In chapter quartics. Thus using the differ but, as concen 14, the treat...
although they are composed of continuous lines. For this reason, we shall call these curves complex. [1748, 2: §. 61]

For instance, the curve corresponding to the equation $y^2 - x^2 = 0$ is complex, since $y^2 - x^2 = (y - x)(y + x)$.

In the *Introductio*, Euler also divided curves into algebraic and transcendental and classified algebraic curves according to the degree of their equations. He gave wide space to conics, which he treated in a general and analytic way. He began his investigation with the general equation of the second degree in two variables

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

and deduced the properties of conic sections from it. Euler highlighted the difference between the *Introductio* and previous books. While previous writers derived the properties of conics from geometric constructions, he examined "what one can deduce from their equation, without recourse to other means" [1749, 2, §. 85]. In order to classify conics, Euler observed that equation (1.4) changed into

$$y^2 = \alpha + \beta x + \gamma x^2$$

by using an appropriate transformation of the coordinate system. He stated that

1) if $\gamma > 0$ and one puts $x = \pm \infty$ then $y^2 = \infty$ and $y = \pm \infty$. Therefore the curve becomes equal to $\pm \infty$ both for $x = +\infty$ and $x = -\infty$. It is a hyperbola.

2) if $\gamma < 0$ and one puts $x = \pm \infty$ then $y^2 = -\infty$ and $y$ is imaginary. Therefore the curve does not become equal to infinity for any value of $x$. It is an ellipse.

3) if $\gamma = 0$ then $y^2 = \alpha + \beta x$ and the nature of the curve does not change according to $\beta > 0$ or $\beta < 0$. If $\beta > 0$ and one puts $x = \infty$ then $y^2 = \infty$ and $y = \pm \infty$. If one puts $x = -\infty$ then $y^2 = -\infty$ and $y$ is imaginary. Therefore the curve becomes equal to infinity only if $x = +\infty$. It is a parabola.

In chapters 9, 10, and 11, Euler investigated and classified cubics and quartics. Then Euler deals with tangents, normals and curvature without using the differential algorithm. In particular he examined cuspidal points, but, as concerns cuspidal points of the second kind in section 333 of chapter 14, the treatment was rather confused. Indeed when he wrote the book he
agreed with de Gua that no algebraic curve could have a cuspidal point of the second kind (de Gua had published a flawed proof of this statement in his [1740]). Later Euler found the following counterexample to de Gua's claim:

$$y^4 - x^3 - 4x^2y - 2xy^2 + x^2 = 0.$$ 

He subsequently wrote a footnote which had to be added to section 333. However, due to a publishing error, the footnote was inserted in the text so that its meaning was unclear. Euler later published a paper to clarify this question (cf. Euler [1749]).

In chapter 21 Euler dealt with transcendental curves, such as $y = x^2$, $y = x^2$ and $y = \log x$. He also considered the equation $y = (-1)^x$ and referred to it as paradoxical because its graph is totally discontinuous: there are pairs of points whose distance is smaller than any assignable quantity and, at the same time, no segment of the straight lines $y = 1$ and $y = -1$ belongs to it (in more modern terms, it is composed of two everywhere dense sets of isolated points) [1748, 2: §517]. On this occasion he briefly mentioned the question of the logarithms of negative numbers. He claimed that the logarithms of negative numbers are imaginary and that this follows from the fact that the ratio $\log(-1) : \sqrt{-1}$ is finite. He observed that if one puts $\log(-n) = a^{15}$, then

$$\log(-n)^2 = \log(n^2) = 2a.$$ 

Therefore both the imaginary number $a$ and the real number $\log(n^2)$ are equal to $\frac{1}{2} \log(-n)$. This statement appears to be contradictory, although Euler stated that the contradiction is only apparent since there exist a given number of infinite logarithms. In order to justify this assertion he considered the equation $a = e^x$ and observed that

$$a = e^x = \sum_{r=0}^{\infty} \frac{1}{r!} x^r.$$ 

According to Euler, equation (1.5) was an equation of infinite degree and had infinite solutions. Therefore there existed infinite logarithms of the number $a$ [1748, 2:§§515–516].

In the *Introductio* Euler also gave an analytical treatment of the transformations of coordinates and used polar coordinates in a systematic way. One particularly interesting solid in space in the

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Euler used the canonical ellipsoid, by sheets (hypers). He did not

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The *Institut* was written as E212 in the preface. In the different chapters, Euler analysis of f
particularly innovative part of the book is the appendix in which Euler examines solid analytical geometry. Similarly for two dimensions, he briefly dealt with surfaces in general and divided them into algebraic and transcendental. In § 91 Euler also gave the rules for the transformation of coordinates in space in the form

\[ \begin{aligned}
    x &= t \cos \zeta + u \sin \zeta \cos \eta - v \sin \zeta \sin \eta - a \\
    y &= -t \sin \zeta + u \cos \zeta \cos \eta - v \cos \zeta \sin \eta - b \\
    z &= u \sin \eta - v \sin \eta.
\end{aligned} \tag{1.6} \]

The investigation of the quadric surfaces was based upon the general equation of the second degree in three dimensions

\[ ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0. \tag{1.7} \]

Euler used equations (1.6) to transform the general quadric surface (1.7) into the canonical equations and then to classify the nondegenerate quadrics into ellipsoid, hyperboloid of one sheet (elliptico-hyperbolica); hyperboloid of two sheets (hyperbolica-hyperbolica), elliptic paraboloid, hyperbolic paraboloid. He did not consider all degenerate quadrics (cf. [1748, 2: §§102–130]). Finally, in chapter 6 of the appendix, Euler also investigated some of the curves which originate from the intersection of surfaces.

§2 Institutiones calculi differentialis

The Institutiones calculi differentialis (later Calculus differentialis, for short) was written before 1748 [see [El]] and published in 1755. It is numbered as E212 in the Eneström Index. It is divided into parts, along with the preface. In the first part of nine chapters, Euler deals with the foundations of the differential calculus. In the second part, which is divided into eighteen chapters, Euler treats various applications of the differential calculus to the analysis of finite and infinite quantities.
§2.1 Infinitesimals and limits

In the preface to the *Calculus differentialis* Euler claimed the absolute correctness of the calculus: “geometric rigour escapes even the slightest error” (Euler [1755, 6].). These words echo Newton’s and Berkeley’s statements: “errors are not to be disregarded in mathematics, no matter how small” (Newton [1704, 334]). Based on this assertion, Berkeley, in his famous *The Analyst*, argued the calculus was not the principle that finite increments were only considered by served that b still commit calculus be b and elements it must have not be based to agree with the crux the equation to Euler, that assuming to lack any b from nothing so the deduct [1755, 6].

Nevertheless, as zeros, ind offers in the they cannot Euler asserts:

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16In §4 of [1734] Berkeley emphasised: “It is said that the minustest errors are not to be neglected in mathematics” and, in §9, he quoted Newton’s Latin words: “Errores quirum

17Since Euler geometry to est.
Analyst, argued that the calculus, in contrast to other branches of mathematics, was not an exact theory. He did not cast any doubt upon the usefulness of the calculus in solving many problems of physics or geometry; nevertheless, he believed that it did not possess solid foundations. In particular, he criticised the principle of cancellation of higher-order differentials (see footnote n.3). In Berkeley’s opinion, by writing \( d(x^2) = (x + dx)^2 - x^2 = 2xdx + dx^2 = 2xdx \), first one considered \( dx \neq 0 \), then one let \( dx = 0 \); thus there was a negation of the first hypothesis and an elementary logical principle was violated. The same criticism had been made of the Newtonian method of first and last reasons: indeed, in the expression \( ((x + h)^n - x^n)/h \), one first considered a finite increase \( h \neq 0 \), but one then let \( h = 0 \) and obtained \( nx^{n-1} \) (cf. Berkeley [1734, §§. 13–18]). Berkeley thought that the calculus achieved correct results only thanks to a compensation of mistakes. This justification was considered by Euler to be insufficient: in the *Calculus differentialis* he observed that by ignoring infinitely small quantities, but not noughts, one could still commit extremely serious errors. Neither could the correctness of the calculus be based on an exhibition of examples in which differential calculus and elementary geometry reached the same results [1755, 6]17. Furthermore it must have appeared completely obvious to Euler that the calculus could not be based on the use of kinematical concepts (in this matter, he was forced to agree with Berkeley’s criticism of Newton (cf. Berkeley [1734, §§. 30–31]).

The crux of the question lay in knowing what meaning to attribute to the equation \( a + dx = a \). The exactness of mathematics required, according to Euler, that the differential \( dx \) should be precisely equal to 0: simply by assuming that \( dx = 0 \), the outrageous attacks on the calculus would be shown to lack any basis. Those who considered infinitesimal quantities as different from nothing could effectively be accused of ignoring geometric rigour and so the deductive processes based on such assumptions were really in doubt [1755, 6].

Nevertheless, it is difficult to base the calculus on differentials understood as zeros: indeed, if we examine the definitions of infinitesimal that Euler offers in the third chapter of his treatise, it can immediately be seen that they cannot be reduced to the mere statement that an infinitesimal is zero. Euler asserts:

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17 Since Euler asserted the independence of analysis from geometry he could not use geometry to establish a solid basis of the calculus.
There is no doubt that every quantity can be diminished until it vanishes completely and is reduced to nothing. But an infinitely small quantity is simply an evanescent quantity and therefore actually equal to zero [1755, 69].

Immediately afterwards, he argues that an infinitesimal can also be defined as a quantity which is less than any assignable quantity (in more modern terms, less than any finite quantity), since a quantity which is less than any assignable quantity is necessarily equal to zero [1755, 69].

The statement that an infinitesimal \( \omega \) or a differential \( dx \) are equal to 0 can be understood in two ways: \( \omega \) is 'numerically' equal to zero (this justifies the principle of cancellation) or else \( \omega \) is a variable quantity 'tending towards zero' (this justifies the existence of different symbols which denote the different way of tending towards zero). From a modern point of view, the difference between the two notions is such that Euler's argument is incomprehensible. Let us imagine, for example, that we subdivide a given quantity \( g \) into two equal parts and then divide one of the halves again, and so on. We have a quantity \( q \) that vanishes; according to Euler's definition, it is an infinitesimal. If we denote the act of vanishing of \( g \) by \( \omega \) and observe that it can analytically expressed as \( q2^{-n} \) or, more simply, as \( 2^{-n} \) (by assuming \( q = 1 \)), then \( \omega = 2^{-n} \) is an infinitesimal \( \omega = 2^{-n} = 0 \). However the meaning of the equality \( \omega = 2^{-n} = 0 \) is unclear: it may be a way (a rather felicitous one) of denoting a limit (therefore \( 2^{-n} = 0 \) would mean \( \lim_{n \to \infty} 2^{-n} = 0 \)), or it may symbolise a equation valid for a process which occurs an actually infinite number of times (therefore \( 2^{-n} = 0 \) would mean \( \text{st}(2^{-n}) = 0 \), where \( n \) is an infinite number).\(^{19}\)

Moreover the definition of the infinitesimal as a quantity smaller than any assignable quantity \( \epsilon \) is vague and does not clarify the question: it leads to the relation \( \omega = 2^{-n} < \epsilon \), but this relation could be understood as \( 2^{-n} < \epsilon \) for every \( \epsilon \) and for a suitable \( n \) depending on \( \epsilon \), or else as \( 2^{-n} < \epsilon \) for every \( \epsilon \) and for \( n \) as a fixed (and therefore infinite) number.

The Eulerian definition seems to confuse various notions (limit, infinitesimal, value of a variable) which, from a modern viewpoint, are clearly different. It might be thought that this situation is due to an imperfect formulation of the definition.

\(^{18}\) Of course, the symbol \( \text{st}(a) \) denotes the standard part of the hyperreal number \( a \).

\(^{19}\) It certainly does not mean that \( 2^{-n} \) is approximately equal to 0 because this is contrary to a literal and substantial interpretation of Euler. The exactness of mathematics makes it necessary that \( a + 2^{-n} = a \) when \( n = \infty \), is a precise equation.

Let \( \omega \) be an infinitesimal, where \( \psi \) is a preceding small num
[1748, 1: §]

In reality, in \( \epsilon \)
He had suggested a potential function that \( z_1 \) and \( z_2 \) following chart by using the important difference between the natural \( w \)

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where \( \omega \) is an infinitesimal, ample, accurate and that \( k \) and \( d \)

value to the infinitely small number of \( \epsilon \)
of the definitions themselves: however, this is an essential characteristic of Euler’s concept, which is actually composed of an inextricable interplay of elements which today appear as different ones. For instance, consider Euler’s derivation of the power series expansions of the exponential and logarithmic functions (see section 1.2). There are some critical steps in it. Indeed Euler used the equality $a^\omega = 1 + \psi$, where $\omega$ and $\psi$ are infinitesimal. In section 114 of the *Introductio* (the first section of chapter 5) he justified it by making a vague reference to what he had stated in the preceding chapter 4:

Let $\omega$ be an infinitely small number, or such small fractions that they are almost equal to nothing, then

$$a^\omega = 1 + \psi,$$

where $\psi$ is an infinitely small number, as well. Indeed, from the preceding chapter, it was established that if $\psi$ was not an infinitely small number, then neither could $\omega$ be an infinitely small number [1748, 1: §.114].

In reality, in chapter 4, Euler had not made any mention of infinitesimals. He had suggested the idea that the difference between $a^{z_1}$ and $a^{z_2}$ of the exponential function $a^x$ might be made equal to a tiny finite quantity, provided that $z_1$ and $z_2$ are taken very close together (at a tiny finite distance). In the following chapter this idea was expressed by the equality $a^\omega = 1 + \psi$, namely by using the language of infinitesimals in an explicit way, as if there were no difference between finite quantities, small as desired, and infinitesimal quantities and if one could move from one language to another in an immediate and natural way, without an adequate theoretical construction.

In a similar way Euler first stated that

$$\omega = \log_a(1 + k\omega),$$

where $\omega$ is an infinitesimal, then he put $a = 10$ and $k\omega = 1/1000000$ and found that $\omega = 0.00000043429$ and $k = 2.30258$ [1748, 1: §.114]. This example, according to Euler, was sufficient to show that $k$ is a finite number and that $k$ depends on the base $a$ of the logarithm. Euler assigned a finite value to the infinitesimal: his reasoning implied that $k\omega$ can assume increasingly small values and that, by reducing the value $k\omega$, one obtained a larger number of exact decimal figures of $k = \log 10$ (the exact value of $k$ can be
obtained when \( \omega \) was evanescent). In Euler’s opinion, an infinitesimal cannot be distinguished, for practical purposes, from a sequence of numbers that becomes very small.

The situation does not change for infinite quantities. Indeed by multiplying \( \omega = \log_a (1 + k \omega) \) by the number \( i \) Euler obtained \( i \omega = \log_a (1 + k \omega)^i \). He observed that the greater the value of \( i \), the more \( (1 + k \omega)^i \) exceeded one, assuming that \( i \) is an infinite number, the value of \( (1 + k \omega)^i \) could be any number greater than one [1748, 1: §.118]. Euler used \( i \) as an infinite number, but the properties of the infinite number \( i \) derived from the fact that the finite number \( i \) increased beyond all limits (and so became infinite): he did not separate the infinite number \( i \) from the process of growth of a finite variable \( i \).

Despite the ambiguity of these procedures, an ambiguity that perfectly corresponds to the ambiguous definition of the infinitesimal, it is well-known that the results obtained are substantially correct and that the Eulerian procedures can be reformulated in forms that are acceptable from a modern viewpoint by using the theory of limits or non-standard analysis. Nevertheless, such a reformulation leads to complex problems.

To make this point clear, let us consider an example which is particularly suited, at least apparently, to being translated into the modern language of limits. In chapter 15 of the *Calculus differentialis*, Euler faced the problem of finding the value of the fraction \( \frac{b - \sqrt{b^2 - x^2}}{x^2} \) for \( x = 0 \). He interpreted 0 as the limit value of a variable quantity \( z \) and, by applying the l’Hôpital rule, found that the value of \( \frac{b - \sqrt{b^2 - z^2}}{x^2} \) at \( x = 0 \) was \( \frac{b - \sqrt{b^2 - z^2}}{z^2} \). [1755, 2: §. 358].

It is completely natural to translate \( \frac{b - \sqrt{b^2 - x^2}}{x^2} \) as \( \lim_{x \to 0} \frac{b - \sqrt{b^2 - x^2}}{x^2} \); however, such a translation of limits produces some straining in the meaning. From a modern perspective, finding the value of \( \frac{b - \sqrt{b^2 - x^2}}{x^2} \) when \( x = 0 \) means that:

a) for every fixed value \( x > 0 \), one considers the function (function in the modern sense of the term) \( f(x) = \frac{b - \sqrt{b^2 - x^2}}{x^2} \) defined for \( x \neq 0 \);

b) the domain of \( f(x) \) has a point of accumulation at 0 so that we can attempt to calculate the limit as \( x \to 0 \);

c) the application of l’Hospital’s rule, under whose hypotheses our case falls, makes it possible to state that such a limit exists and is equal to \( \frac{1}{2b} \).

In this p a function, \( \lambda = \lim_{x \to 0} f(x) \) any \( \epsilon > 0 \) the \( |f(x) - \lambda| < \) associates w function \( f(x) \) a discontinuity

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is equal to \( \frac{1}{2b} \). 

d) finally, we defined a new function \( F(x) \), which will be continuous at

the point 0, by setting

\[
F(x) = \begin{cases} 
\frac{b - \sqrt{b^2 - x^2}}{x^2} & x \neq 0 \\
\frac{1}{2b} & x = 0.
\end{cases}
\]

In this procedure we use notions such as limit, value and extension of

a function, whose meaning is opportunely and explicitly defined. Indeed,

\[ \lambda = \lim_{x \to c} f(x) \], where \( f(x) \) is a function with domain \( D \) in \( R \), means: given

any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( x \) belongs to \( D \) and \( |x - c| < \delta \), then

\[ |f(x) - \lambda| < \epsilon. \] By \( \lambda = f(c) \) we intend: \( \lambda \) is the number that the function \( f \)

associates with the number \( c \). If \( \lim_{x \to c} f(x) \) exists and is equal to \( \lambda \), while the

function \( f(x) \) is not defined at the point \( c \) or \( f(c) \neq \lambda \), we can remove the

discontinuity at \( c \) by defining the new function \( x \to \begin{cases} f(x) & x \in D - \{c\} \\
\lambda & x = c. \end{cases} \)

These definitions presuppose knowledge of the notions of set, real

numbers, function in the modern sense, continuity, etc.: it is such concepts that

enable real analysis to give a clear and rigorous formulation of the intuitive

notion of one quantity approaching another. However, Euler did not have the

mathematical concept of set, nor the theory of real numbers nor the modern

notion of function (see Ferraro [2004]). In such a situation, it was impossible

to give a precise definition of limit and distinguish between the extension

of a function, limit process and assignment of a value to a function: it was

only possible to refer to a vague intuition of a process whereby a quantity \( A \)

approaches a quantity \( B \). In contrast to other eighteenth century mathematicians,

Euler generally avoided even the use of the term 'limit'.\(^{20}\) Furthermore,

he never considered the general case of a variable quantity \( A \) which goes to

any finite quantity \( B \), but only variables that vanish or endlessly increase.\(^{21}\)

Even though Euler based the calculus on the notions of 'vanishing', 'evanes-

cent quantity', ..., he never offered a definition, not even a vague or imprecise

one, of these terms. The idea of an evanescent quantity seems to be more a

notion borrowed from the natural world than from mathematical notions.

\(^{20}\) As far as I am aware, only in the preface to the *Calculus differentialis* [1755, 7] did

Euler use 'limit' to mean "approaching a limit".

\(^{21}\) Euler seems to believe that every other case of nearing a limit can be included within

these two cases (see Euler [1732–33, 44] and Fuss [1843, 2:229–231]).
The mental image that may be associated with it is that of a physical entity (such as the quantity of gunpowder, in the initial example of the Calculus differentialis) which we can consider to an increasingly smaller extent or else, still remaining within the field of mathematics, the image of a segment which increasingly diminishes until it becomes a single point and disappears as a segment.

Euler's use of the evanescent quantity corresponds to Euclid's use of the 'common notion': a part is smaller than the whole. In the Elements this is not an axiom in the modern sense of the term but a general and rather vague principle where use is made of the terms 'part', 'smaller' and 'whole' without their meaning being defined (not even implicitly). In the same way, Euler regarded the fact that a quantity vanishes or increases infinitely as simply part of the idea of quantity and not in need of further explanation or clarification: it was thought to be a notion which formed part of common human knowledge.

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Now let us return to Euler's proof of the expansion \( a^x = \sum_{r=0}^{\infty} \frac{1}{r!} (kx)^r \). In this proof, Euler considers \( i \) as an infinite number; nevertheless, he justified the relation \( \frac{i-1}{i} = 1 \) by an intuitive, direct consideration of the process of growth of a finite variable \( i \). He stated:

However much larger the number that we substitute for \( i \), the more the value of the fraction \( \frac{i-1}{i} \) comes close to unity. Therefore, if \( i \) is a larger number than any assignable one, the fraction \( \frac{i-1}{i} \) equals unity (Euler [1748, 124]).

The equation \( \frac{i-1}{i} = 1 \) was obtained by first interpreting \( i \) as a finite variable and then as an infinite number. It was an approximate equation if \( i \) was finite; nevertheless, when \( i \) was an infinite number, it needed to be considered as a precise equation, equivalent to the statement that the infinitesimal was precisely zero.

This example helps to clarify the difference between Eulerian infinitesimals and non-standard analysis. Nowadays a hyperreal number \( a \) has a standard part \( b \), from which it differs by an infinitesimal (in symbolic form, \( b = st(a) \) or \( a \approx b \)). In particular, one obtains \( st \left( \frac{i-1}{i} \right) = 1 \), an equation which we can regard as the non-standard version of \( \lim_{i \to \infty} \frac{i-1}{i} = 1 \). According
to Euler, however, the formula $a + dx = a$ had to be an exact equation, not an approximate one: his objective was precisely to eliminate approximations such as $\frac{1}{i} \approx 1$. He refused the idea of basing the calculus on infinitesimals as actually existent entities since they implied errors of approximation.

There are other reasons that make the assimilation of Euler’s calculus into non-standard analysis problematic. Today infinitesimal and infinitely large numbers are used in non-standard analysis as elements of a set $\star R$: they are elements of a rich and well-organised algebraic structure which encodes how a sequence approaches a limit. A possible construction of $\star R$ is the following. Let $m$ be a finitely additive measure on the set $N$ of the positive integers such that: for all $A \subseteq N$, $m(A)$ is 0 or 1; $m(A) = 0$ if $A$ is finite, $m(N) = 1$.\footnote{For a proof of the existence of this measure, see Lindström [1988, 84–85].}

Then, consider the equivalence relation $\sim$ on the set $S$ of all sequences of numbers

$$\{a_n\} \sim \{b_n\} \text{ iff } m\{n : a_n - b_n = 0\} = 1,$$

The set $\star R$ of the hyperreal numbers is defined by $\star R = S/\sim$. The classes of equivalence of the sequences $\{0\}, \{1/n\}, \{1/n^2\}, \{n\}, \{n^2\}, \{n^3\}$ are six elements of $\star R$; the first three are infinitesimals, the last three are infinite numbers.

The intuitive idea of approaching a limit is at the basis of this construction of $\star R$; however, in no case can a demonstration concerning objects of $\star R$ refer to such an intuition but only to the axioms and the definitions which enable us to arrive at $\star R$. The same is true for the calculation of the limits in the set of real numbers $R$, which entirely depends on the definition of limit and the construction of the set $R$.

Viewed from the perspective of the refined concepts of the twenty-first century, the way in which Euler codifies the idea of approaching is naive and over-simplified: an infinite number is simply an short way of indicating a variable that goes to infinity while an infinitesimal is a short way of indicating a variable that goes to zero. The rules that Euler uses upon infinite and infinitesimal numbers constitute an immediate extrapolation of the behaviour of a finite variable $i$ tending to $\infty$ or to 0: what is wholly missing is the complex construction of $\star R$ and the assumptions upon which it is based.

Rather than making use of sophisticated twenty-first century concepts, the Eulerian use of infinite numbers and infinitesimals can be better understood if they are thought of as ‘fictions’: infinite numbers and infinitesimals
are fictions of variable quantities, represented by symbols with which one operates in analogy with 'real' quantities. Leibniz had also made use, in certain occasions, of the notion of fiction in order to justify infinitesimals; he used it as the basis for using imaginary quantities and other entities introduced in mathematics.

The use of fictions made Eulerian mathematics extremely different from modern mathematics. A fiction presupposes the existence of real objects to which it relates by imitating them. The mathematics of fictions distinguishes between true and false objects; the latter express qualities or characteristics of true ones. Today, infinite numbers and infinitesimals, as elements of \(^\ast\mathbb{R}\), are not ontologically different from other elements of \(^\ast\mathbb{R}\), which all derive from the construction of \(^\ast\mathbb{R}\). According to Euler, an infinitesimal quantity was not an actual quantity in the true sense of the term but a short way of indicating how 'true' quantities vanish. Analogously, in modern mathematics, 0, as an element of the set \(\{0, 1, 2, \ldots\}\), is a number that exists in the same way as the other numbers 1, 2, 3, \ldots; in Euler's opinion, however, 0 was the symbol that represented the absence of quantity; it was not a quantity in the true sense of the term but a fiction that could be treated as a quantity. There was an ontological difference between the fictions and the 'real' objects of Eulerian mathematics, a difference which does not exist in modern mathematics (on this concept, see Ferraro [2004]).

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In the *Calculus differentialis* Euler does not provide a geometric interpretation of infinitesimals but does this in his *Institutionum calculi differentialis Sectio III*\(^{23}\). Euler (see fig.2) considers the finite increments \(Pp = \Delta x\), \(nn = \Delta y\) of the variables \(AP = x\) and \(y = PM\), and imagines that the point \(p\) continually approaches \(P\) (or rather, in Eulerian language, \(Pp\) becomes infinitely small or reduces to nothing). He states that:

A) when \(p\) meets \(P\), \(Pp\) will form the differential \(dx\).

B) the figure cannot represent infinitesimal quantities so that an appeal is made to the fiction that \(Pp\) does not so much represent itself, so to speak, but an infinitesimal part of itself \(dx\).

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\(^{23}\)Euler intended this work to form the third part of the treatise on the differential calculus where the geometric applications of the calculus were dealt with; it was published posthumously in 1862 (see Euler [OP]).
C) the element of the infinitesimal curve $Mm$ can be considered as a small straight line: indeed, when the arc $Mm$ is infinitely small, it coincides with the straight line $Mm$ since the difference between the arc and the segment becomes smaller than any pre-established quantity.

![Figure 2: Finite increments.](image)

In other words, the infinitesimals $Pp$ and $Mm$ are simultaneously considered as a way of indicating a limit (evanescent quantities), null (or pairs of coincidental points to use modern terminology), but also as elements of the lines. It follows that the limit $P$ reached by $p$ is indistinguishable from the limit process $p \to P$ and both are confused with the infinitesimal trait $pP = dx$: this infinitesimal part can be treated as a fiction [OP, 336–338].

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A further remarkable aspect of the Eulerian notions of infinite and infinitesimal quantities is the meaning that could be attributed to the equation ‘$0 = 0$’ and ‘$\infty = \infty$’. Let us consider the relation

$$\frac{c}{a} + \frac{c}{a+b} + \cdots + \frac{c}{a+(i-1)b} = C + \frac{b}{c} \log(a+ib)$$

(2.1)
where \( i \) is an infinite number and \( C \) is a finite constant. The relation (2.1) essentially states that the two quantities behave in the same way when \( i \) increases unlimitedly. Comparing infinite and infinitesimal quantities meant for Euler assessing their behaviour when they simultaneously increased unlimitedly or vanished. This idea is at the basis of the distinction between arithmetic and geometric equations, by which Euler aims to provide an improved mathematical explanation of the equation between infinitesimals \((0 = 0)\) and between infinite numbers \((\infty = \infty)\).

In the *Calculus differentialis* Euler stated that, given two quantities \( a \) and \( b \), the equation \( a = b \) can be understood in an arithmetic sense (namely, \( a = b \), if \( a - b = 0 \)) and in a geometric sense (\( a = b \), if \( a/b = 1 \)). While the arithmetic equality coincides with the geometric one for finite quantities, the situation is different for infinite numbers and infinitesimals. For infinitesimals, the arithmetic equality, which is always verified, does not imply the geometric one: in Euler’s symbolism, \( 0 = 0 \) does not imply \( 0/0 = 1 \). For infinite numbers, the geometric equality does not imply the arithmetical one, that is \( \infty/\infty = 1 \) does not imply \( \infty = \infty \).

By using this distinction, Euler believed he could justify the principle of cancellation. He observed that \( a \pm ndx = a \) \((n\) being any number) is true in that it is not only verified in an arithmetical sense \(((a \pm ndx) - a = ndx, \text{being}\ ndx = 0)\) but also in a geometric sense. Indeed, one obtains \((a \pm ndx)/a = 1\) and this means that the infinitesimals vanish before any finite quantity. The situation is analogous for higher-order infinitesimals. Euler stated that “the infinitely small quantity \( dx^2 \) vanishes before \( dx \), since the quantities \( dx + dx^2 \) and \( dx \) \((\text{both evanescent:} \ dx + dx^2 = dx = 0)\) go to zero in the same way \((dx + dx^2) : dx = 1 + dx = 1\). More generally, if \( m < n \), then \( adx^m + bdx^n \) is equal to \( adx^m \) because the arithmetical and geometrical equalities

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24 Euler proved (8) in [1734–35a].

25 For example, the following theorem should be interpreted in such a way: the sum of the series of the reciprocals of prime numbers \( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \) is infinitely large but is nevertheless infinitely smaller than the harmonic series \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \); the sum of the first is equal to the logarithm of the sum of the second. Since \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \) is equal to \( \log \infty \), one has \( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \log(\log \infty) \) (cf. Euler [1737, 242-243]).

26 Euler showed how the laws of elementary arithmetic can be adapted to the case \( 0 = 0 \). If one considered \( n : 1 = 0 : 0 \) and interpreted it as \( n \cdot 0 = 0 \). As a consequence, \( 0/0 \) could be equal to any number \( n \) [1755, 70]. Naturally the example does not show us how \( 0/0 \) becomes exactly equal to a certain \( n \). This can only be understood by considering the zero as the limit of a variable.
\[\frac{\partial x}{\partial y} + b dx^n = ax^m = 0 \text{ and } \frac{\partial x^m + b dx^n}{\partial x^m} = 1 + \frac{a}{b} \frac{dx}{x} = 1\]

are verified (cf. Euler [1755, p.70 and p.74]).

We could briefly say: if \(A = B + C\), and if \(C\) goes to zero before \(B\) (in other words, \(A/(B + C) = 1\)), then \(A = B\). In modern terms, \(A\) and \(B\) are asymptotically equal\(^2^7\) or have the same asymptotic behaviour.\(^2^8\) Nevertheless, while it is fairly clear what is meant by the ratio \(\psi/\xi\) between infinitesimal and infinite quantities \(\psi\) and \(\xi\), Euler does not clarify how it is possible to move from asymptotic equations to algebraic equations, or rather how the quantities \(\psi\) and \(\xi\), which are asymptotically equal, can be manipulated as though they were equal numbers.

Naturally there is no intention of attributing to Euler modern asymptotic notions or the definition of the Peano derivative\(^2^9\): this would be as difficult as stating that Euler uses infinitesimals in the sense of non-standard analysis. I merely noted that Eulerian notions possessed many facets which, with the hindsight of modern knowledge, we can appreciate but which were hazy to Euler: thus the idea of evanescent quantity contained subtleties that lead us to think of the standard notion of limit or of modern infinitesimals or of asymptotic processes or generalised limit concepts. I would not be surprised if traces of other modern notions are found in Eulerian infinitesimals. Nevertheless, one cannot really speak of confusion between different aspects, since Euler could not confuse notions that did not yet exist. Instead, he refers to a

\(^{27}\)Two functions \(f(x)\) and \(g(x)\) are said to be asymptotically equal as \(x \to c\) (\(c\) is a finite or infinite point of the set on which the functions \(f(x)\) and \(g(x)\) are defined) if there exists a function \(h(x)\) such that \(\lim_{x \to c} h(x) = 1\) and \(f(x) = h(x)g(x)\) in some neighbourhood of the point \(c\) (except possibly at \(c\) itself). If \(g(x)\) does not vanish in some neighbourhood of \(c\), this condition is equivalent to \(\lim_{x \to c} f(x) = 1\).

\(^{28}\)In particular, the idea of the approach of two quantities, which we can see here in operation, rather than the usual definition of limit, makes us think of the following generalised definition: The function \(\lambda(x)\) is the limit of the function \(f(x)\) as \(x\) approaches \(c\) if for any positive number \(\epsilon\), however small, there is some positive number \(\delta\), which in general depends on \(\epsilon\), such that \(|f(x) - \lambda(x)| < \epsilon\) whenever \(|x - c| < \delta\). Indeed, when a quantity \(A\) was conceived of as approaching another quantity \(B\), it could occur that both quantities \(A\) and \(B\) were variable.

\(^{29}\)Let there exist a number \(\delta > 0\) such that for all \(|x| < \delta\), one has \(f(x + c) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \delta(x)\), where \(a_0, a_1, a_2, \ldots, a_r\) are constants, \(h(x) \to 0\) as \(x \to 0\) and \(h(0) = 0\). Then \(a_r\) is called the Peano derivative of order \(r\) of the function \(f(x)\) at the point \(c\).
primordial idea of approaching (which we can term ‘proto-limit’) from which later mathematics has derived various modern notions.

§2.2 The nature of the differential calculus

The above-described conception of differentials allowed Euler to consider the calculus as a calculus of functions (in Euler’s sense). In the Calculus differentialis he stated that the differential calculus was not concerned with investigating differentials, which are nothing, but with defining their mutual ratio, which had a determinate value [1755, 5]. He claimed that functions were the genuine subject-matter of the differential calculus and that differentials were mere tools for dealing with functions. For example, given the function \( y = x^2 \) whose differential is \( dy = 2xdx \), the calculus studied the differential coefficient \( \frac{dy}{dx} = 2x \) and not the differential \( 2xdx \) (cf. Euler [1768–1770, 1:6]).

In the preface to the Calculus differentialis, Euler provided an example where the calculation of the differential coefficient seems to look forward to the modern definition of the derivative. He considered the function \( y = x^2 \) and observed that \( y(x + \omega) : \omega = (2x + \omega) : 1 \) (where \( y(x + \omega) \) is the increment, \( 2x + \omega^2 \), of \( x^2 \)), so that the smaller \( \omega \) is, the closer \( (2x + \omega^2) : \omega \) becomes to \( 2x : 1 \), even though it actually reaches \( 2x : 1 \) only when the increment has completely vanished [1755, 5].

The similarity with the modern notion of derivate must not deceive us. In chapters 3 and 4 of the Calculus differentialis, where the rules of the calculus are formulated, the introduction of differential coefficients was considerably more tortuous. Indeed, Euler tried to include the analysis of infinites organically into the method of finite differences, considering it as a special case of this method “which occurs when the differences, which were previously supposed to be finite, are taken infinitely small.” [1755, 1: §114]. The idea of limit, which had already been expressed with sufficient clarity in the preface to the Calculus differentialis, was thus absorbed by the notion of infinitely small.

Euler had defined the finite differences of any order\(^{30} \) in chapter 1, where he had set out the rules of the sum and the product of finite differences

\(^{30}\text{Set } y^{(n)} = y(x + n\omega), \text{ for a nonnegative integer } n, \text{ and } y = y^{(0)}, \text{ Euler defined } \\
\Delta y = y^{(1)} - y, \quad \Delta y^{(n)} = y^{(n+1)} - y^{(n)}, \quad \Delta^m y = \Delta^{m-1} y^{(1)} - \Delta^{m-1} y, \quad \Delta^m y^{(n)} = \\
\Delta^{m-1} y^{(n+1)} - \Delta^{m-1} y^{(n)}, \text{ for } m > 1 \text{ and } n > 0 \text{ (cf. } [1755, 1: §§1-7]).}
and had calculated the differences of algebraic functions and exponential, logarithmic, trigonometric functions. The investigation of the finite differences of these functions led him to state (cf. [1755, 1: §§8–24]) that \( \Delta y, \Delta^2 y, \Delta^3 y, \ldots \) could be expressed in the form

\[
\Delta y = P \omega + Q \omega^2 + R \omega^3 + S \omega^4 + \cdots \tag{2.2}
\]

\[
\Delta^2 y = P \omega^2 + Q \omega^3 + R \omega^4 + \cdots, \tag{2.3}
\]

\[
\Delta^3 y = P \omega^3 + Q \omega^4 + R \omega^5 + \cdots, \tag{2.4}
\]

\[\ldots\]

In chapter 1, Euler also posed the inverse problem of finding the quantity \( y \) that generates a given finite difference \( z \). He termed the quantity \( y \) the sum of \( z \) and used the symbol \( \Sigma \) to denote the inverse operation of \( \Delta \) (namely \( y = \Sigma z(x, \omega) \) meant that the difference \( \Delta y \) was equal to \( z(x, \omega) \)). According to him, the sum \( y = \Sigma z \) was not unique but was of the type \( y = \Sigma x + C \), where \( C \) was an arbitrary constant. For instance, the sums of \( \Delta y = a \omega \), \( \Delta y = \omega^2 \), and \( \Delta y = \omega^3 \) are \( \Sigma a \omega = a y + C \), \( \Sigma \omega^2 = \omega y + C \), and \( \Sigma \omega^3 = \omega^2 y + C \), respectively (cf. Euler [1755, 1: §§25–36]).

In chapter 4, Euler stated that the differential calculus originated by letting \( \omega = dx \) be infinitesimal in (2.2); in this way \( \Delta y \) also became an infinitesimal and, by ignoring higher-order infinitesimals (which vanish before \( \omega \)) one obtained \( \Delta y = P \omega \), which could be written as \( dy = P dx \). According to him, this reasoning proved that the differential \( dy \) of a function could be expressed by means of a certain function \( P \) of \( x \) multiplied by the differential \( dx \) and that the ratio between \( dx \) and \( dy \) was a finite number (Euler [1755, 1: §§112–113]).

If one takes into account the example given in the preface, Euler could have defined the differential coefficient as

\[\text{[Notes:}\]

\[\text{31} \text{However it should be noted that if } y = \Sigma z \text{ is the sum of } z = \Delta y, \text{ the most general sum is not } y = \Sigma x + C \text{ but } y = \Sigma x + f(\omega), \text{ where } f(\omega) \text{ is a periodic function that assumes the same value at the points } x + \omega, x + 2\omega, x + 3\omega, \ldots, \text{ such as } C \cos \frac{2\pi x}{\omega}, \text{ for fixed } x \text{ and } \omega (\text{Euler could make such considerations as one can deduce from his work [1750–51], in particular 465–467). Assuming that the sum is of the type } y = \Sigma x + C, \text{ with constant } C, \text{ means that Euler does not reason on a fixed set of points } \{x + \omega, x + 2\omega, x + 3\omega, \ldots\} \text{ but upon the general or abstract quantities } \omega \text{ and } x \text{ (for any } \omega); \text{ namely his procedure holds for the variable } \omega \text{ and } x, \text{ not for the specific values that } \omega \text{ and } x \text{ assume.}
\]

\[\text{[Notes:}\]

\[\text{32} \text{See also the example given in Euler [1765, 80].}\]
\[
\frac{\Delta y}{\omega} = \frac{P \omega + Q \omega^2 + R \omega^3 + S \omega^4 + \cdots}{\omega} = P + Q \omega + R \omega^2 + S \omega^3 + \cdots
\]

for \( \omega \) as an evanescent quantity. However, Euler preferred to give a direct definition of the differential \( dy \) of the function \( y(x) \) (by means of \( \Delta y = P \omega + Q \omega^2 + R \omega^3 + S \omega^4 + \cdots \) and only an indirect definition of the differential coefficient. Thus, for example, in order to determine the differential of \( y = x^n \), he put \( dy = y^{(1)} - y = (x + dx)^n - x^n \) and applied the binomial theorem. He obtained

\[
dy = y^{(1)} - y = (x + dx)^n - x^n = nx^{n-1}dx + \frac{n(n-1)}{1 \cdots 2} x^{n-2}dx^2 + \cdots
\]

By neglecting higher-order infinitesimals, which vanish before \( dx \), he had \( dx^n = nx^{n-1}dx \) [1755, 1: §152].

One should observe a rather peculiar situation. Even though the calculus dealt with finite quantities \( dy/dx \), in the actual construction of the calculus, Euler considered \( dy/dx \) as the real ratio of the differentials \( dy \) and \( dx \). Euler used differentials on the basis of the fact that the differential \( dy \) of the function \( y(x) \) could be considered as the increment of \( y(x) \) when \( x \) has an infinitesimal increment \( dx \), namely on the basis of the formula \( dy = y(x + dx) - y(x) \). Thus, in the calculation of the differential of \( x^n \), there is no reference to limit process and evanescent quantity, in contrast to the example presented in the preface to the Calculus differentialis. Despite these ambiguities, Euler really did place differential coefficients at the centre of attention and made a systematic attempt to reduce the calculus to a calculus of finite quantities.

The situation does not differ significantly from that regarding higher-order differentials. These were regarded as particularly problematic from the origins of the calculus and were, for example, the object of an attack by Nieuwentiijdt upon Leibniz. In chapter 4 of the Calculus differentialis, Euler stated that second-order differentials originated by letting \( \omega = dx \) in (2.3). Since the terms \( Q \omega^3, R \omega^4 \ldots \) vanished in comparison with \( P \omega^2 \), one obtained \( d^2y = Pdx^2 \), where \( dx^2 \) was the square of \( dx \) [1755, 1: §125]. Analogously, higher-order differentials derived from higher-order finite differences \( \Delta^n y = P \omega^n + Q \omega^{n+1} + R \omega^{n+2} + \cdots \).

Euler stated that the second differential was simply the differential of the first differential (since the second difference was simply the difference
of the first difference) [1755, 1: §126]. Nevertheless, stating that \( d^2y \) was simply \( d(dy) \) led to the question as to what is meant by differentiating \( dy \). Indeed, it involved making something become evanescent when it was already evanescent. In order to justify this, Euler assumed that the variable quantity \( x \) received equal increments, or rather that the sequence of values \( x, x^{(1)} = x + dx, x^{(2)} = x + 2dx, \ldots, x^{(n)} = x + ndx \) was assigned to the variable \( x \) [1755, 1: §124].\(^{33}\) In this way, however, \( d^2x \) was constantly equal to zero (in Euler’s terminology, the higher-order differentials \( d^2x, d^3x, d^4x, \ldots \) were per se equal to zero): \( d^2x \) vanished in a different sense from \( dx \) and the intuitive meaning of a differential understood as an evanescent variable disappeared. Euler therefore attempted to provide a different interpretation of the equation \( d^2x = 0 \). If one applied (2.3) to the function \( y = x \), then \( P = Q = R = \cdots = 0 \), that is \( d^2y = d^2x = 0dx^2 + 0dx^3 + \cdots \) and \( P = Q = R = \cdots = 0 \); it was possible to state that the second differential \( d^2x \) vanished before \( dx^2 \) and all the higher powers of \( dx \) [1755, 1: §125]. Therefore, \( d^2x \) was both zero per se and a relative zero (namely, \( d^2x \) was actually equal to zero and was a quantity that vanished before \( dx^2, d^3x, d^4x, \ldots \)). A similar interpretation could be given for \( d^3x, d^4x, \ldots, \ldots \) it is also worth pointing out that, if it is true that the finite second difference \( \Delta^2y \) is by definition equal to \( \Delta(\Delta y) \), it does not appear at all obvious that the second differential, defined by means of (2.3), must coincide with the differential of \( dy \). In other terms, given a function \( y = y(x) \), whose first and second differentials are \( dy = pdx \) and \( d^2y = qdx^2 \), there is no a priori guarantee that the second differential of \( y \) coincided with the differential coefficient of \( p \). In order to prove this, Euler set \( dp = qdx \) and obtained \( ndp = nqdx \), where \( n \) represented any constant quantity. By letting \( n = dx \) (therefore \( dx \) is constant), he obtained \( dpdx = qdx^2 \) and \( d^2y = d(dy) = d(pdx) = qdx^2 + 0 = qdx^2 \), namely [1755, 1: §127]. The reasoning can be repeated; if \( dq = rdx \), then \( d^3y = dq = rdx^3 \), if \( dr = tdx \), then \( d^4y = dr = tdx^4, \ldots \); therefore the higher-order differentials of \( y \) can be calculated one after another by differentiating \( p, q, r, t, \ldots \).

It is natural to ask ourselves why Euler did not directly define the second differential as \( d^2y = qdx^2 \), where \( q = dy/dx \). Yet even in the preface to the

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\(^{33}\)It is possible to think the sequence \( x^{(n)} = x + ndx \) as a sequence \( x^{(n)} = x + n\omega \), where the increment \( \omega \) is initially finite and subsequently made evanescent; \( dx=\text{constant} \) means that the increment diminishes in the same way for all the points \( x^{(n)} = x + ndx \). Whatever value is assigned to \( dx \) tending uniformly to zero, we will always have \( d(dx) \) constantly = 0.
Calculus differentialis, Euler seemed to suggest precisely this definition for $d^2y$. Indeed, he had observed that since infinitesimals were equal to zero, the higher-order differentials were never considered per se but in relation to each other. More precisely, given a function $y = f(x)$, whose differential coefficient is a certain function $p$, the second differentials were obtained by considering the ratio of increment of the function $p$ with other increments and the symbols of the differentials serve only to give a convenient representation of certain finite quantities [1755, 8].

Instead, in chapter 4 of the Calculus differentialis, Euler made use of the complex construction described above to introduce higher-order differential coefficients: he seemed to be worried about the confrontation with the Leibnizian tradition and to demonstrate how Leibniz' and Bernoulli's differential calculus was capable of being translated into the new calculus of functions and their differential coefficients.

The Leibnizian calculus was not based on functions but on curves analytically expressed by an equation $f(x, y) = 0$. The independent variable was not chosen a priori and therefore it was not established a priori that $dx$ was a constant. The notion of a function is based rather on a clear distinction between dependent and independent variable: the entire Eulerian construction, beginning with the formulas of finite differences, was made using the hypothesis that $x$ was the independent variable. Euler emphasised that the increments of $x$ were taken as constants (that is, $x$ was the independent variable and $d^2x = 0$) and that he was studying the variability of functions in relation to $x$. In the case that constant increments were not assumed, the second differentials have a more complicated expression. Indeed, if $dx$ was not constant, then the differential of $p(x)dx$ was derived by considering the increment $dx$ of $x$ and $d^2x$ of $dx$. Put $p(x + dx) = p(x) + q(x)dx$, one has

$$d^2y = d(dy) = d(pdx) = p(x + dx) \cdot (dx + d(dx)) - pdx = (p + qdx) \cdot (dx + d^2x) - pdx = pdx^2 + qdx^2 + qdx^2 - pdx.$$  

Since $qdx^2$ vanished before $qdx^2$, he obtained $d^2y = pdx^2 + qdx^2$ [1755, 1: §§.128–129].

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34In chapter 1, Euler had already noted that, as regards finite differences, the first difference $\Delta y = y(x_1) - y(x) = y(x + \omega) - y(x)$ was not influenced by the sequence $x_n$ while the second differences changed according to the nature of $x_n$. This remark is also valid for differentials. On this topic, see Bos [1974, 25–31].
In chapter 4 Euler restricted himself to observing that nothing can be said with certainty about second differentials if $dx$ is not assumed to be constant: there is no explanation about how $d^2x$ can be reduced in this case to the theory of evanescent quantities. He dealt with the question more thoroughly in chapter 8, where he applied the rules of differentiation to functions such as $V = (dx^2 + dy^2)^{1/2}$ and $V = ydy/dx$ by considering $dy$ and $dx$ as variables. For instance, he found that the differential of $V = (dx^2 + dy^2)^{1/2}$ was $dV = \frac{dx^2 + dy^2}{\sqrt{dx^2 + dy^2}}$ and that the differential of $V = ydy/dx$ was $dV = dx + \frac{yd^2x}{dy} - \frac{ydx^2}{dy^2}$ [1755, 1: §.250].

Euler observed that the formulae containing higher-order differentials have a vague meaning since they do not possess any determined value per se but assume values which vary according to which differential is taken as constant [1755, 1: §. 251]. For example, given the expression $\frac{yd^2x}{dxdy}$, if one considered $dx$ as a constant then the differential was $\frac{yd^2y}{dxdy}$, while if $dy$ was a constant, then one obtained a different result

\[
\frac{yd^2x}{dxdy}.
\]

Furthermore, Euler believed that higher order differentials did not have an effective use in analysis [1755, 1: §.263]: they could be eliminated due to the differential coefficients $p, q, r, \ldots$, (defined by the relations $dy = pdx$, $dp = qdx$, $dq = rdx, \ldots$); namely, by putting $d^2y = qdx^2$, $d^3y = dq = rdx^3, \ldots$. For instance, Euler stated that $\sqrt{dx^2 + dy^2}$ as a constant was often found in the applications of the calculus. Set $dy = p(x)$ and $dp = q(x)$, he obtained $dx \sqrt{1 + p^2}$ as constant, $d^2x \sqrt{1 + p^2 + \frac{pdx^2}{\sqrt{1 + p^2}} = 0}$, $d^3x = -\frac{pdx^2}{1 + p^2}$, and $d^2y = qdx = qdx^2 + pdx = qdx^2 + \frac{dx^2}{1 + p^2} = \frac{pdx^2}{1 + p^2}$ (cf. Euler [1755, 1: §.269]).

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An important clarification must be made in order to understand the Eulerian calculus. Differentiation was not a pointwise defined operation; in other words, the differential was not defined at a specific point $x_0$ of the domain of the function. Euler never considered limitations to the domain of

\[\text{(continued)}\]
variables: such a concept was completely absent from his mathematics and only considered general or abstract quantities. Differentiation was regarded as a global operation that involved the entire analytical expression \( y(x) \) as was aimed at determining another analytical expression \( p(x)dx \). For example, once he had established that \( d(\log x) = dx/x \), Euler also did not hesitate to apply this formula to the case where the variable is imaginary, without making any distinction between real and imaginary variable. Thus he found the differential \( dy = \frac{dx}{\sqrt{1-x^2}} \) of the function \( y = \frac{1}{\sqrt{1-x^2}} \) and asserted that the differential of an imaginary quantity could be real [1755, 1. §§182–183]. More explicitly, in his [1749] he had stated: “For as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that \( d(\log x) = dx/x \), whatever value we give to \( x \), either positive, negative or even imaginary, we would never able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains” [1749, 143–144]. This is a clear formulation of the so-called principle of the generality of algebra, one of the cornerstones of Euler’s formal methodology.

§2.3 Two applications of the differential calculus

In the second part of the *Calculus differentialis* Euler shows several applications of the differential algorithm. In particular, he gave wide room to the determination of the maxima and minima, to the theory of series, to the numerical determination of the roots of equations. Here I limit myself to dealing with the Taylor theorem and the Euler-Maclaurin sum formula. These theorems played a crucial role in Euler’s theory of series: indeed, he viewed the Taylor series as a general method for developing a function; while the Euler-Maclaurin sum formula provided a general method for summing a numerical series.

In order to derive the Taylor series Euler\(^{38}\) observed that

\[
\begin{align*}
y(x + dx) &= y + dy \\
y(x + 2dx) &= y + dy + dy + d^2y = y + 2dy + d^2y
\end{align*}
\]

\(^{37}\)We saw that the concept of continuity was also global (cf. section 1.3). On local and global viewpoints in Euler’s analysis see Ferraro [2000].

\(^{38}\)See also [1736b].
\[ y(x + 3dx) = y + 3dy + 3d^2y + d^3y \]

\[ \ldots \]

\[ y(x + ndx) = y + ndy + \frac{n(n-1)}{1 \cdot 2} d^2y + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d^3y + \ldots \]

Euler considered \( n \) to be an infinite number. By putting \( n = \omega/\omega \), he obtained the Taylor series

\[ y(x + \omega) = y(x) = \frac{\omega dy}{dx} + \frac{\omega^2 d^2y}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \ldots \ [1755, 2: \S.44]. \]

In chapter 5 of the *Calculus differentialis* Euler gave the following derivation of the Euler-Maclaurin sum formula\(^{39}\), which he wrote in the form

\[ (2.5) \quad S_n = \int zd^x + \frac{1}{2} z + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} \frac{d^{2n-1}z}{dx^{2n-1}} \quad \text{(with } S(0) = 0), \]

where \( S_n = \sum_{i=1}^{n} z_i \) is the partial sum of the series \( S_n = \sum_{i=1}^{n} z_i \), \( z(x) \) is a function\(^{40}\) such that \( z_i = z(i) \) and \( B_n \) are the Bernoulli numbers\(^{41}\). Euler [1755, 2: \S.103–122] interpreted the operation \( S \) of partial sum as a symbolic operation that enjoys the following formal properties:

a) finite and infinite additivity, which he explicitly formulated in these terms: if \( y_x = p_x + q_x + r_x + \ldots \), then \( S_y = S_p + S_q + S_r + \ldots \), that is,

\[ \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} p_n + \sum_{n=1}^{\infty} q_n + \sum_{n=1}^{\infty} r_n + \ldots, \]

b) the commutativity of the operations \( S \) and \( \frac{d^n}{dx^n} \): \( \frac{d^n}{dx^n} (Sy) = S \frac{d^n}{dx^n} y \).\(^{42}\)

Euler set \( v = y(x - 1) = y_{x-1} = v_x \ (x = 2, 3, \ldots) \) and \( v(1) = A \) to get

\[ S_v = \sum_{n=1}^{\infty} y_n = A + \sum_{n=2}^{\infty} y_{n-1} = A + S_y - y. \]

By applying the additivity of the

\(^{39}\)Different derivations are found in other papers (see Ferraro [1998]).

\(^{40}\)In Euler’s opinion, this function exists since a sequence is an analytical expression.

\(^{41}\)There are several definition of Bernoulli numbers \( B_n \). Here I refer to the following:

\[ B_{-1} = 1 + \sum_{i=1}^{\infty} (-1)^{(r/2)+1} \frac{B_r}{r!} \tau'(\lfloor r \rfloor < 2\pi, \lfloor x \rfloor \text{ is the integral part of } x), B_0(x) = 1, \text{ which is closer to Eulerian use}. \]

\(^{42}\)From a modern viewpoint this formula only has a meaning if we interpret \( S \) as an integral. In this case, it corresponds to \( \frac{d^n}{dx^n} \int y \, dx = \int \frac{d^n}{dx^n} y \, dx \).
Euler's treatises on infinitesimal analysis:

operation $S$ to $v = y(x - 1) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{(n)}}{n! dx^n}$ and rearranging it, he derived

$$S \left( \frac{dy}{dx} \right) = y - A + \sum_{n=2}^{\infty} (-1)^n S \frac{d^n y}{n! dx^n}$$

and

$$S(z) = \int z dx + \sum_{n=1}^{\infty} (-1)^{n+1} S \frac{d^n y}{(n+1)! dx^n}$$

(where $z = \frac{dy}{dx}$ and the condition $S(0) = 0$ holds). By differentiating and applying property 2, Euler found

$$S \frac{d^h z}{dx^h} = \frac{d^{h-1} z}{dx^{h-1}} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \cdot S \frac{d^{h+n-1} z}{dx^{h+n-1}}, \quad h = 0, 1, 2, \ldots \left( \text{here } \frac{d^{-1} z}{dx^{-1}} = \int z dx \right).$$

He then expressed $S z$ as $S z = \int z dx + \sum_{n=1}^{\infty} a_n \frac{d^{n-1} z}{dx^{n-1}}$ and derived (2.5) (cf. Euler [1755, 2: §§ 103–122]).

§3 Institutionum calculi integralis

The *Institutionum calculi integralis* (later, *Calculus integralis*, for short) was published in three volumes in 1768, 1769, and 1770. In December 1763 the work had been finished for many months and was presented to the St. Petersburg Academy on August 7, 1766 (see Fuss [1843, 1:671] and Eneström [EI]). The three volumes are numbered as E342, E366 and E385, respectively, in the Eneström Index.

The first book consists of an introduction and three sections. In the introduction, Euler discusses the nature of the integral calculus. In the first section of nine chapters Euler deals with the integration of elementary functions; in particular he examines approximate integration and integration by series and infinite products. In the second section which contains seven chapters, Euler investigates differential equations and elliptic integrals. The third section, which is not divided into chapters, is devoted to differential equations involving higher-order differentials and transcendental functions.
Sectio prima. De integratione formularum differentialium

1. De integratione formularum differentialium rationalium.
2. De integratione formularum differentialium irrationalium.
3. De integratione formularum differentialium per series infinitas.
4. De integratione formularum logarithmicarum et exponentialium.
5. De integratione formularum angulos sinusve angulorum implicantium.
6. De evolutione integralium per series secundum sinus cosinusve angulorum multiplo-
   rum progredientes.
7. Methodus generalis integralis integralla quacunque proxime invenienda.
8. De valoribus integralium, quos certis tantum casibus recipiunt.
9. De evolutione integralium per producta infinita.

Sectio secunda. De integratione aequationum differentialium

1. De separatione variabilium.
2. De integratione aequationum differentialium ope multiplicatorium.
3. De investigatione aequationum differentialium, quae per multiplicatores datae for-
   mae integrabiles reddantur.
4. De integratione particulari aequationum differentialium.
5. De comparatione quantitatum transcendentium in forma $\int \frac{Pdx}{\sqrt{Ax^2+Bx+C}}$ con-
   tentarum.
6. De comparatione quantitatum transcendentium in forma $\int \frac{Pdx}{\sqrt{Ax^2+Bx+C+Dx^3+Ex^4}}$
   contentarum.
7. De integratione aequationum differentialium per approximationem.

Sectio tertia. De resolutoae aequationum differentialium magis complicatarum

De resolutione aequationum differentialium, in quibus differentialia ad plures dimensiones
assurgent vel adeo transcendentes implicatur.

The second book is divided into two sections. The first section of twelve chapters is devoted to the integration of second-order differential equations. The second section of five chapters dealt with differential equations of third and higher order.
4. De aequationibus differentio-differentialibus, in quibus altera variabilis uniam habe
dimensionum.
5. De integratione aequationum differentialium secundi gradus, in quibus altera vari-
abiles unam dimensionem non superat, per factores.
6. De integratione aliarum aequationum differentio-differentialium per idoneos multi-
plicatores instituenda.
7. De resolutione aequationis \( ddy + ax^nydx^2 = 0 \) per series infinitas.
9. De transformatione aequationum differentio-differentialium hujus formae \( Npdfx^2 + 
Mddy + Lddy = 0 \).
10. De constructione aequationum differentiodifferentialium per quadraturas curvarum.
11. De constructione aequationum differentio-differentialium ex eorum resolutione per 
series infinitas petita.
12. De aequationum differentiodifferentialium integracione per approximationes.

Sectio secunda. De resolutione aequationum differentialium tertii altiorumque 
graduum, quae duas tantum variabiles involvunt
1. De integratione formularum differentialium tertii altioris\^\textsuperscript{e} gradus simplicium.
2. De resolutione aequationum hujus formae: \( Ayy + B \frac{dy}{dx} + C \frac{d^2y}{dx^2} + D \frac{d^3y}{dx^3} + E \frac{d^4y}{dx^4} + \text{etc.} \) = 
0 sumto elemento \( dx \) constante.
3. De integratione aequationum differentialium hujus formae \( X = Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + 
D\frac{d^3y}{dx^3} + \text{etc.} \).
4. Applicatio methodi integrandi in capite praecedenti traditae ad exempla.
5. De integratione aequationum differentialium hujus formae \( X = Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + 
D\frac{d^3y}{dx^3} + \text{etc.} \).

The third book of *Calculus integralis* consists of two parts, along with an 
appendix and a *Supplementum*. The first part, which contains three sections 
with six, five, and three chapters, respectively, and the second part, which 
contains four chapters, deal with partial differential equations. The appendix 
contains seven chapters and is devoted to the calculus of variations\(^\text{43}\). In the 
*Supplementum* Euler discusses some particular differential equations.

\(^{43}\)On Euler’s calculus of variations, I refer to Fraser [1994].
Institutionum calculi integralis volumen tertium, in quo methodus inveniendi functiones duarum et plurium variabilium, ex data relatione differentialium cujusvis gradus pertrectatur. Una cum appendice de calculo variationum et supplemento, evolutionem casuum prorsus singularium circa integrationem aequationum differentialium continente.

Pars Prima. Seu investigatio functionum duarum variabilium ex data differentialium cujusvis gradus relatione

Sectio prima. Investigatio duarum variabilium functionum ex data differentialium cujusvis gradus relatione
1. De natura aequationum differentialium, quibus functiones duarum variabilium determinantur in genere.
2. De resolutione aequationum, quibus altera formula differentialis per quantitates finitas utcunque datur.
3. De resolutione aequationum, quibus binarum formularum differentialium altera per alteram utcunque datur.
4. De resolutione aequationum, quibus relatio inter binas formulas differentiales et unicam trium quantitatum variabilium proponitur.
5. De resolutione aequationum, quibus relatio inter quantitates \( \frac{dx}{dx} \), \( \frac{dy}{dy} \), et binas trium variabilium \( x, y, z \), quaecunque datur.
6. De resolutione aequationum, quibus relatio inter binas formulas differentiales \( \frac{dx}{dx} \), \( \frac{dy}{dy} \), et omnes tres variabiles \( x, y, z \), quaecunque datur.

Sectio Secunda. Investigatio duarum variabilium functionum ex data differentialium secundi gradus relatione
1. De formulis differentialibus secundi gradus in genere.
2. De una formula differentiali secundi gradus per reliquas quantitates utcunque data.
3. Si duae vel omnes formulæ secundi gradus per reliquas quantitates determinantur.
4. Alia methodus peculiaris hujusmodi aequationes integrandi.
5. Transformatio singularis earundem aequationum.

Sectio tertia. Investigatio duarum variabilium functionum ex data differentialium tertii altiorumque graduum relatione
1. De resolutione aequationum simplicissimarum unicam formularum differentialem involventium.
2. De integratione aequationum altiorum per reductionem ad inferiores.
3. De integratione aequationum homogenearum, ubi singuli termini formulas differentiales ejusdem gradus continent.

Pars altera. Investigatio functionum trium variabilium ex data differentialium relatione
1. De formulis differentialibus functionum tres variabiles involventium.
2. De inventione functionum trium variabilium ex dato cujuspiam formulae differentiales valore.
3. De resolutione aequationum differentialium primi gradus.
4. De resolutione aequationum differentialium homogenearum.
Appendix de calculo variationum.
1. De calculo variationum in genere.
2. De variatione formularum differentialium duas variabiles involventium.
3. De variatione formularum integralium simplicium duas variabiles involventium.
4. De variatione formularum integralium complicatarum duas variabiles involventium.
5. De variatione formularum integralium variabiles involventium, et duplicem relationem implicantium.
6. De variatione formularum differentialium tres variabiles involventium, quarum relationio unica aequatione continetur.
7. De variatione formularum integralium, tres variabiles involventium, quarum una est functio binarum reliquarum spectatur.

Supplementum continens evolutionum casuum singularium circa integrationem aequationem differentialium

§3.1 The problematic concept of integration as an antiderivative

Euler defined integration as anti-differentiation; in other words, he regarded integration as the inverse operation of differentiation by which one returned from the differential to the function generating the differential [1768, 1: §1–2]. This definition is rather problematic. First, Euler did not provide any proof of the existence of the anti-differential of a given function. Second, Eulerian functions consisted only of elementary functions and their composition (cf. section 1.1) and, therefore, it can be argued that the definition of the integral should be interpreted in the following sense: given an elementary function \( f(x) \), the integral \( \int f(x)dx \) is a quantity \( F(x) \) such that \( dF = f(x)dx \).

Euler was aware that many simple functions could not be integrated by means of elementary functions. He tackled the question in the introduction to the Calculus integralis, where he compared integration with inverse arithmetical operations [1768–70, 1: §3 and 29]. He stated that analytical operations are always opposed in pairs. In the same way as addition is opposed to subtraction, multiplication to division or the raising to a power to the extraction of the root, differentiation and integration are also opposed. In certain cases, the inverse operations of subtraction, division and extraction of a root could not be performed and led to new ‘quantities’, namely negative, rational and irrational numbers. Similarly, integration was not always successful in finding the function generating the differential. In such cases, integration led to new transcendental ‘quantities’.
Some remarks are appropriate. Firstly, Euler explicitly rejected the Leibnizian notion of integral as the sum of infinite infinitesimals. Indeed, when he treated the approximate integration of a function \[1768-70, 1:\$8, 297-304\], Euler expressed the integral \( y(x) = b + \int_a^x X(x)dx \) in the form

\[
y(x) = b + \int_a^x X(x)dx = b + aX(a) + X(a + \alpha) + X(a + 2\alpha) + \cdots + X(a + n\alpha), \text{ where } x = a + n\alpha.
\]

Equation (3.1) could be assumed as the definition of an integral, taking \( \alpha \) as an infinitesimal. However, (3.1) was precise only if one considered infinitesimals as nothing; in this case equation (3.1) offered a useful rule for the approximate calculation of integrals. According to Euler, the concept of the integral as the sum of an infinite number of infinitesimals was similar to the concept of lines as aggregates of infinite points. Both could be admitted (or better tolerated), as long as there was reference to the true principles of the calculus and geometry,\(^4\) thus avoiding every kind of sophism \([1768-1770, 1:3, 302]\). In other words, it was possible to take into account non-null infinitesimals but it constituted only an imprecise and approximate version of the notion of integral, which nevertheless had useful applications. Naturally, if one imagined \( \alpha \) as an evanescent quantity, it might have been possible to follow a definition of a integral which was acceptable from the Eulerian point of view. However, Euler was not interested in such a possibility and regarded the definition of an integral as an antiderivative as satisfactory.

Secondly, there is a profound difference between the way Euler perceived inverse operations and how they are perceived today. Today, when an operation \( T: O \in S \rightarrow A \in V \), transforming an object \( O \) into an object \( A = T(O) \), is given and we consider the inverse operation \( I_T \) such that \( I_T(A) = O \), first we must be sure that given an object \( A \in V \) there exists an element \( O \in S \) such that \( I_T(A) = O \), then we can operate on \( O = I_T(A) \). If the object \( O = I_T(A) \) does not exist, first we must construct an appropriate set of objects \( S^* \) including \( S \) and such that \( I_T(A) \) exists, for every \( A \), then we can operate on the new object \( I_T(A) \).

Instead Euler did not construct the appropriate class of the objects \( S^* \) which made the inversion of the operation \( T \) always possible. When the

\[^4\text{In the case of integration, the true principle was the concept of the integral as an antiderivative.}\]
object $I_T(A)$ did not exist, he considered the undefined symbol $I_T(A)$ as a formal object that had to satisfy the condition $T(I_T(A)) = A$. This condition allowed him to manipulate $I_T(A)$. Euler therefore operated on the unknown object $I_T(A)$ as if it was known. The scope of his research is solely to make the object $I_T(A)$ known\textsuperscript{46}. Moreover, in most cases the operation $T$ and its converse $I_T$ had an immediate interpretation in geometric or arithmetical terms. This interpretation assured that the manipulation of $I_T(A)$ was not a mere formal game\textsuperscript{46}.

In the case of integration, if the function $F(x) = \int f(x)\,dx$ such that $dF = f\,dx$ did not exist (i.e., it was not one of the elementary functions), one handled the symbol $\int f(x)\,dx$ subject to the condition $d(\int f(x)\,dx) = f\,dx$, and so determined the properties of the unknown object $\int f(x)\,dx$. The geometric interpretation guaranteed that the formal entity $\int f(x)\,dx$ had a meaning.

The situation was similar for negative numbers, rational numbers, and radicals, as Euler wrote in the Calculus integralis, and in other cases as well. For instance, according to Euler’s definition of the sum of a series, the operation of the sum $\sum(f_n)$ of a function series $f_n$ is the inverse operation of the operation $\delta(f)$ of development of a function $f$ (see section 1.2). If $f(x)$ does not exist (i.e., it is not one of the known functions), we can handle the symbol $\sum f_n(x)$ subject to condition $\delta(\sum f_n(x)) = \sum f_n(x)$. If $\sum f_n(x)$ is a convergent\textsuperscript{47} series, it has a immediate, numerical meaning (even if the sum is unknown).

Thirdly, in the Calculus integralis Euler stated that if $\int f(x)\,dx$ was not an elementary function, then it is a new transcendental quantity or function. At the first glance, one could think that Euler changed the point of view of the Introductio and that he used a more general concept of a function. In effect if we look at the Calculus differentialis and Calculus integralis, we observe that that, apart from elementary functions, Euler used the word ‘function’ to denote three types of non-elementary quantities: 1) quantities that were analytically expressed by an integral $\int f(x)\,dx$ or a differential equation, 2)\textsuperscript{46}I note a certain similarity with the classical analytical method in Pappus’s sense (one operates upon unknown objects subject to appropriate conditions).\textsuperscript{46}In some cases (imaginary numbers and divergent series) the inverse operation had no intuitive interpretation and its meaning derived from establishing a formal connection between mathematical objects. For instance, Euler’s formula $e^{ix} = \cos x + i\sin x$ (with $i = \sqrt{-1}$).\textsuperscript{47}On divergent series, see Ferraro [C].
discontinuous quantities, 3) inexplicable quantities⁴⁸ (in the remainder of this article I will refer to them as non-elementary transcendental functions). However, they were not considered as true functions. Indeed Euler made a distinction between the different type of quantities which were treated in analysis. On the one side there were algebraic quantities and those quantities (logarithmic, exponential and trigonometric quantities) that could “be treated in the same easy way as algebraic quantities” [1780, 522]. These quantities played a special role in analysis and were the only functions in the proper sense of the term. Instead, (relations between) quantities that could not be expressed using elementary functions were not to be placed on the same plane as algebraic functions: they were not functions in the strict sense of the term (see [1768–70, 1:14]).

One of the main reason of the different status between functions is to be seen in the fact that functions in the proper sense of the term were thought to satisfy the following conditions (see Ferraro [G]):

(C1) A special calculus concerning these functions existed (i.e., a group of algorithmic rules related to the analytical expression, such as the rule of the calculus of trigonometric functions).

(C2) The values of basic functions were considered as given since they could be calculated by performing algebraic operations and using tables of values.

(C1) and (C2) were precisely those conditions that allowed the object ‘function’ to be accepted as the solution to a problem. In general, it is necessary to exhibit a known object in order to solve a problem. In analysis, an object was considered to be known if one could manipulate it and if one could calculate its values with sufficient approximation. During the eighteenth century only elementary functions were thought to be known objects to the point that they could be accepted as the final solution to a problem. Instead, relations between quantities, different from elementary functions, did not satisfy conditions (C1) and (C2)⁴⁹.

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In the first section of book 1 of the Calculus integralis, Euler systematically discusses integrals of the type \( \int f_{a,b}, \ldots (x) \, dx \), where \( f_{a,b}, \ldots (x) \) is

⁴⁸ will discuss discontinuous functions below (section 3.3). On inexplicable functions, see Ferraro [1998].

⁴⁹ For a more detailed discussion of this complex question, I refer to Ferraro [2000b], [G] and [ST].
a specific and determinate elementary function that can depend on one or more parameters \(a, b, \ldots\), however, his conception of integration and functions leads to some initial difficulties even by chapter 2, which is devoted to the integration of irrational functions. Here Euler shows that \(\int x^{m-1}(a + bx^n)^{\frac{e}{n}} dx\) can be integrated by means of elementary functions only if \(\frac{m}{n}\), or \(\frac{e}{n}\) or \(\frac{m}{n} + \frac{e}{n}\) are integer numbers [1768, 1: §104 and 160]. In the other cases, the integral gives rise to “a new kind of transcendental function” and, when this occurs, one could only “endeavour to assign it a value near to the true one” [1768, 1: §93].

Euler devised an approach to the investigation of non-elementarily integrable functions which can be divided into the following four parts:

1) broad classes of integrals \(\int X_s dx\), where \(X_s\) was a non-integrable function depending on one or more parameters, were to be specified;

2) for every class \(\int X_s dx\), a special integral \(\int X dx\) belonging to the class \(\int X_s dx\), was to be identified;

3) the special integral \(\int X dx\) was to be investigated in order to obtain a known function, namely a function that satisfied conditions (C1) and (C2);

4) all the integrals of the class \(\int X_s dx\) were to be reduced to the integral \(\int X dx\).

This approach led to the investigation of several special functions in the *Calculus integralis*, although Euler did not consider the knowledge obtained as completely satisfactory – mainly with regard to the observance of the preceding point 3) – and thus none of the new transcendental quantities were considered to be a new function in the strict sense of the term. For instance, when Euler faced the problem of the integration of \(\int \frac{x^{m-1}}{\log^n x} dx\), he observed:

These integrations \(\int \frac{x^{m-1}}{\log^n x} dx\) depend on formula \(\int \frac{x^{m-1}}{\log x} dx\). Put \(x^m = z\), hence \(x^{m-1} dx = (1/m) dz\) and \(\log x = (1/m) \log z\), this formula is reduced to the very simple form \(\int \frac{dz}{\log z}\). If the integral of this kind could be assigned, it should be of a very wide use in analysis...It therefore seems that this formula \(\int \frac{dz}{\log z}\) furnished a peculiar type of transcendental functions, which however merits a more careful investigation.” [1768, 1: § 219].

Euler noted that the integrals \(\int \frac{x^{m-1}}{\log^n x} dx\) are connected to the integrals
\[ \int \frac{ax}{x} \, dx \text{, which depend on } \int \frac{a \, dx}{x} \text{. By integrating the expansion} \]

\[ \frac{a^x}{x} = \frac{1}{x} + \sum_{r=1}^{\infty} \frac{1}{r!} (\log a)^r x^{r-1} \]

term by term, one obtains

\[ (3.2) \quad \int \frac{a^x \, dx}{x} = \log x + \sum_{r=0}^{\infty} \frac{1}{r!} (\log a)^r x^r, \]

which becomes \( \int \frac{e^x \, dx}{x} = C + \log x + \frac{x}{1} + \frac{1}{2} \frac{x^2}{2!} + \frac{1}{3} \frac{x^3}{3!} + \cdots \) for \( a = e \). The transformation \( e^x = z \) yields

\[ (3.3) \quad \int \frac{dz}{\log z} = C + \log \log z + \frac{\log z}{1} + \frac{1}{2} \frac{\log^2 z}{2!} + \frac{1}{3} \frac{\log^3 z}{3!} + \cdots \] \[ \text{[1768–70, 1: §. 228].} \]

According to Euler, if we assume that the integral \( \int \frac{dz}{\log z} \) is real for \( 0 < z < 1 \), since \( \log(\log z) \) is imaginary for \( 0 < z < 1 \), then the constant \( C \) is imaginary, therefore \( \int \frac{dz}{\log z} \) is imaginary for \( z > 1 \). Vice versa, if we assumed that \( \int \frac{dz}{\log z} \) is real for \( z > 1 \), since \( \log(\log z) \) is real for \( z > 1 \), then \( C \) is real and \( \int \frac{dz}{\log z} \) is imaginary for \( z < 1 \). That led Euler to state that the nature of this function was not known enough [1768–70, 1: §.228].

§3.2 Selected topics from the Calculus integralis

In the Calculus integralis a theory of definite integration was lacking\(^{51}\), however Euler calculated many definite integrals. When he sought the value of a definite integral \( \int_{a}^{b} f(x) \, dx \), first, he considered the infinite integral \( F(x) = \int f(x) \, dx \) and determined the arbitrary constant under the condition \( F(0) = 0 \), then he calculated the value of \( F(x) \) for \( x = t \).\(^{52}\) One of

\[ ^{50} \text{Formula (3.3) contains an inaccuracy (depending on the integration of } 1/x).} \]

\[ ^{51} \text{Mascheroni proved that if } 0 < z < 1, \text{ then } \int \frac{dz}{\log z} = C + \log(-\log z) + \frac{\log z}{1} + \frac{1}{2} \frac{\log^2 z}{2!} + \frac{1}{3} \frac{\log^3 z}{3!} + \cdots \]

and the constant \( C = 0.577 \ldots \) is the Euler constant (see Mascheroni [1790–1792]).

\[ ^{52} \text{A theory of definite integration is found in his [1785].} \]

\[ ^{53} \text{In some papers (see, e.g., [1785]) Euler denoted the definite integral } \int_{a}^{b} f(x) \, dx \text{ by the symbol } \int_{a}^{b} f(x) \, dx \text{.} \]

\[ \begin{bmatrix} \text{a} & \text{b} \\ a & b \end{bmatrix} \]

(\text{the Latin words "ab" and "ad" mean "from" and "to"). Therefore, we might write: } \begin{bmatrix} \int \text{f(x)dx} \\ \text{from } x = a \text{ to } x = b \end{bmatrix} \]
the most remarkable integrals Euler examined was \( \int_0^1 x^{p-1}(1-x^n)^{\frac{q}{n}} \, dx \). In chapter 9, he expounded a lot of interesting results and, in particular, the expansion of the integral into infinite products. Euler also introduced the symbol \( \left( \frac{p}{q} \right) \) to denote it \([1768-70, 1:§.375]\). By replacing \( x^n \) by \( y \) in the equation \( \left( \frac{p}{q} \right) = \int_0^1 x^{p-1}(1-x^n)^{\frac{q}{n}} \, dx \), one obtains \( \left( \frac{p}{q} \right) = \frac{1}{n} \int_0^1 y^{\frac{p}{n}}(1-y)^{\frac{q}{n}} \, dy \), therefore Euler’s symbol \( \left( \frac{p}{q} \right) \) is connected with the beta function \( B(x, y) \) by the equation \( \left( \frac{p}{q} \right) = \frac{1}{n} B \left( \frac{p}{n}, \frac{q}{n} \right) \). Euler demonstrated several properties of the symbol \( \left( \frac{p}{q} \right) \). For instance

\[
\left( \frac{p}{q} \right) = \left( \frac{q}{p} \right) = \frac{p+q}{pq} \frac{n(p+q+n)}{(p+n)(q+n)} \frac{2n(p+q+2n)}{(p+2n)(q+2n)} \ldots
\]

\[
\left( \frac{n}{q} \right) = \left( \frac{q}{n} \right) = \frac{1}{q}, \quad \left( \frac{p}{n} \right) = \frac{1}{n}
\]

\[
\left( \frac{p}{n-p} \right) = \left( \frac{n-p}{p} \right) = \int_0^1 \frac{x^{p-1}}{\sqrt{(1-x^n)^p}} \, dx = \frac{\pi}{n \sin \frac{\pi}{n}}
\]

(3.4) \[
\left( \frac{p}{q} \right) \left( \frac{p+q}{r} \right) = \left( \frac{q}{r} \right) \left( \frac{q+r}{p} \right) = \left( \frac{p+r}{q} \right) \left( \frac{q+r}{p} \right)
\]

In particular he gave the value 1, 2, 3, 4, \ldots to \( p, q, r \) in formula (3.3) and obtained the following table

---

\(^{53}\)Euler had previously published most of these results (see Ferraro [ST]).

\(^{54}\)The beta function is defined by the formula \( B(\xi + 1, \zeta + 1) = \int_0^1 x^\xi (1-x)^\zeta \, dx \).
Elliptic integrals are certainly one of the most interesting topics of the second section of book 1. In [1768–70, I: §, 579] Euler observed that the formula cannot be integrated by means of elementary functions, however it is possible to investigate the integral \( \int \frac{Pdx}{\sqrt{A+2Bx+Cx^2+2Dx^3+Ex^4}} \). In the special case \( \frac{dx}{\sqrt{A+Cx^2+E}} \), Euler put \( y = \frac{b\sqrt{A+Cx^2+E}}{A-Ex^2} \) and obtained

\[
\frac{dx}{\sqrt{A+Cx^2+E}} + \frac{dy}{\sqrt{A+Cy^2+E}} = 0.
\]

He denoted the integral \( \int_0^x \frac{dx}{\sqrt{A+Cx^2+E}} \) by the symbol \( \Pi : x \) (I will write \( \Pi(x) \)). The integration of equation (3.5) yields

\[
\Pi(x) + \Pi(y) + c = 0.
\]
Euler's treatises on infinitesimal analysis:...

Euler then put \( x = 0 \) in equation (3.6). Since \( y(0) = b \) and \( \Pi(0) = 0 \), it follows that \( \Pi(0) + \Pi(b) + c = 0, c = -\Pi(b) \), and

\[
\Pi(x) + \Pi(y) = \Pi(b).
\]

He then put \( x = p, y = q, b = -r \), and obtained \( \Pi(p) + \Pi(q) + \Pi(r) = 0 \), where \( r, p, \) and \( q \) satisfy the equation

\[
(A - Ep^2 q^2)r + q\sqrt{A(A + Cp^2 + Ep^4)} + \sqrt{A(A + Cq^2 + Eq^4)} = 0.
\]

By putting \( q = p \) he found the duplication formula

\[
\Pi(r) = 2\Pi(p) \quad \text{for} \quad r = \frac{2p\sqrt{A(A + Cp^2 + Ep^4)}}{A - Ep^4} \quad [1768-70, 1: \S\S.612-613].
\]

He then generalized this result and stated \( \Pi(z) \) is equal to \( n\Pi(p) \) for appropriate values of \( z \) and \( p \) and studied the integral formulas \( \int_0^z \frac{L + Mz^2 + Nz^4}{\sqrt{A + Cz^2 + Ez^4}} dz \), \( \int_0^z \frac{dx}{\sqrt{A + 2Bx + Cx^2 + 2Dx^3 + Ex^4}} \) and \( \int_0^z \frac{a + bx + cx^2 + dx^3 + ex^4}{\sqrt{A + 2Bx + Cx^2 + 2Dx^3 + Ex^4}} dx \).

***

Most of the Calculus integralis is devoted to differential equations\(^{35}\). In the second section of book 1, Euler dealt with first-order differential equations. In particular, he explained the method of the separation of variables and proved that the solution to the general first-order linear equation \( dy + Pydx = Qdx \) is

\[
y = e^{-\int Pdx} \int e^{\int Pdx} Qdx.
\]

He also dealt with several equations which can be transformed into equations with separable variables. For instance, he demonstrated that the equation \((a + bx + cy)dx = (d + ex + gx)dy\) could be transformed into an equation with separable variables by posing \( a + bx + cy = t \) and \( d + ex + gx = u \). In [1768-70, 1: \$443\] he applied the theorem on mixed differential (see section 3.3) to prove that the differential equation \( Pdx + Qdy = 0 \) could be integrated if

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.
\]

\(^{35}\)It is worthwhile noting that in the Calculus integralis Euler did not deal with multiple integration. (He illustrated the notion of double integral in his [1769].)
He then showed that if the equation \( Pdx + Qdy = 0 \) was not integrable, one could transform it into an integrable equation by multiplying it by an appropriate quantity \( M \) \([1768-70, 1: \S.459]\).

Euler termed the integral of a differential equation containing an adequate number of arbitrary constants as complete integral (in modern terms, general integral). He gave the name particular integral to a relation between the variables that satisfies the equation and that does not contain any new constant quantity (see [1768-70, 1: \S.540]). Euler observed that the complete integral yields infinite particular integrals, however there were particular integrals that are not contained in the complete integral\(^{56}\) This problem was puzzling. Euler had already tackled it in a paper significantly entitled Exposition de quelques paradoxes dans le calcul intégral (Exposition of some paradoxes of the integral calculus), where he had showed that the equation

\[
(y - x \frac{dy}{dx})(y + (2a - x) \frac{dy}{dx}) = c^2
\]

had a solution which was not contained in its general integral. In the Calculus integralis, Euler explained the notion of singular integral by using the following example:

\[
\frac{dx}{dy} = \sqrt{a - x}. 
\]

He observed that \( x = a \) is a solution to equation (3.7) but it is not contained in the general solution \( a - x = \frac{1}{4} (C - y)^2 \). This example shows that singular solutions were sometimes associated with singularities of \( dy/dx \) and Euler tackled the problem of establishing in which cases \( x = a \) was a singular solution of the equation \( dy = dx/Q(x) \) with \( Q(a) = 0 \) (cf. [1768–1770, 1: \S.547]). He realized that if the solution to a differential equation is obtained using an integrating factor \( M(x) \), then the singular solution may arise when \( 1/M(x) = 0 \) (cf. Archibald [2003, 334–335]). Euler also provided a criterion to distinguish the singular solution from a particular integral without knowledge of the general solution.

In the second book of the Calculus integralis Euler discussed ordinary differential equations of second and higher order. In particular he considered the equation

\[
A_0 y + A_1 \frac{dy}{dx} + A_2 \frac{d^2y}{dx^2} + \cdots + A_n \frac{d^ny}{dx^n} = 0, 
\]

\(^{56}\)Euler did not use the name 'singular integrals' and his definition of particular integrals is more general than the modern one.
where $A_0, A_1, \ldots, A_n$ are constants. By putting $y = e^{\lambda x}$ he derived the algebraic equation

$$A_0 + A_1 \lambda + A_2 \lambda^2 + \cdots + A_n \lambda^n = 0.$$  

If this equation has $n$ distinct solutions $\lambda_1, \ldots, \lambda_n$, then $y = e^{\lambda_1 x}, \ldots, y = e^{\lambda_n x}$ are $n$ particular solutions and the general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$. Euler also considered the cases in which the roots are complex or multiple. For instance, he showed that if $\lambda_1 = \lambda_2$, then $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are two particular solutions and the general solution is $y = c_1 e^{\lambda_1 x} + c_2 xe^{\lambda_1 x} + c_3 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$ (cf. [1778-70, 2: §1117–1120]).

Euler then gave a method for solving the nonhomogeneous linear equation with constant coefficients

$$A_0y + A_1 \frac{dy}{dx} + A_2 \frac{d^2y}{dx^2} + \cdots + A_n \frac{d^n y}{dx^n} = X(x).$$

For instance, in case of a second-order equation

$$Ay + B \frac{dy}{dx} + C \frac{d^2y}{dx^2} = X(x),$$

he multiplied the equation by $e^{\alpha x} dx$ and obtained

$$\int \left( e^{\alpha x} Ay + e^{\alpha x} B \frac{dy}{dx} + e^{\alpha x} C \frac{d^2y}{dx^2} \right) dx = \int e^{\alpha x} X(x) dx. \quad (3.9)$$

The left-hand side is equal to $e^{\alpha x} \left( A'y + B' \frac{dy}{dx} \right)$ for appropriate values of $\alpha, A'$, and $B'$. By comparing $\int \left( e^{\alpha x} Ay + e^{\alpha x} B \frac{dy}{dx} + e^{\alpha x} C \frac{d^2y}{dx^2} \right) dx$ and $e^{\alpha x} \left( A'y + B' \frac{dy}{dx} \right)$ Euler derived

$$B' = C$$

$$A' = B - \alpha C,$$

$$A' = \frac{A}{\alpha}.$$

This system enabled him to determine $\alpha, A'$, and $B'$. Consequently equation (3.9) can be written in the form

$$A'y + B' \frac{dy}{dx} = e^{-\alpha x} \int e^{\alpha x} X(x) dx.$$
To solve this first-order equation Euler used the integrating factor \( e^{\mu x} \) and showed that \( \mu \) was one of the two solutions to the equation \( A - Bt + Ct^2 = 0 \), the other being \( \alpha \) (cf. [1768–70, 2: §1138–1163]).

***

When a differential equation did not possess a closed-form solution Euler often sought a series solution. For instance, in chapter 8 of the second book of the Calculus integralis [1768–1770, 2: §§967–976], he tried a series solution of the differential equation

\[
(3.10) \quad x^2(a + bx^n) \frac{d^2 y}{dx^2} + x(c + ex^n) \frac{dy}{dx} + (f + gx^n) = 0.
\]

He put

\[
(3.11) \quad y = \sum_{j=0}^{\infty} A_j x^{\lambda+jn} \quad (A_0 \neq 0)
\]

and substituted (3.11) into (3.10) to obtain

\[
\beta_0 A_0 + \sum_{j=1}^{\infty} (\alpha_{j-1} A_{j-1} + \beta_j A_j) x^{\lambda+jn} = 0,
\]

where

\[
\begin{align*}
\hat{a}_0 &= \hat{e}(\hat{e} - 1)a + \hat{e}a + \lambda f, \\
\hat{a}_0 &= \hat{e}(\hat{e} - 1)b + \hat{e}b + \hat{e}c \\
\hat{a}_j &= jn(jn + 2\hat{e} - 1)b + jn\alpha + \hat{a}_0 \quad \text{(for } j > 0) \\
\beta_j &= jn(jn + 2\hat{e} - 1)a + jn\beta \quad \text{(for } j > 0).
\end{align*}
\]

Euler applied the principle of undetermined coefficients and found

\[
\hat{e}(\hat{e} - 1)a + \hat{e}a + f = 0 \text{ and } \hat{a}_j A_j = \hat{a}_{j-1} A_{j-1} \quad \text{(for } j > 0).
\]

Then he obtained the values of \( \lambda \) by solving the equation \( \lambda(\lambda-1)a + \lambda c + f = 0 \) and then determined \( A_j \) (for \( j > 0 \)) by recurrence (he chose \( A_0 \) arbitrarily).  

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\(^{57}\)Euler had published his results on differential equations with constant coefficients in his [1743] and [1750].
Euler’s treatises on infinitesimal analysis: ...

Euler observed that if the equation $\lambda(\lambda - 1)a + \lambda c + f = 0$ had two different solutions, $\lambda_1$ and $\lambda_2$, one found two series of the form (3.11), which furnished the general integral of (3.10). If $\lambda(\lambda - 1)a + \lambda c + f = 0$ had only one root or the difference between the two values of $\lambda$ was divisible by $58 n$, the general integral could not be expressed as the sum of two series of the form (3.11). Euler showed that in this case the general integral had the form

$$y = \log x \sum_{j=0}^{\infty} A_j x^{\lambda_1+jn} + \sum_{j=0}^{\infty} B_j x^{\lambda_2+jn} + \sum_{j=0}^{\infty} C_j x^{\lambda_1+jn},$$

where $\lambda_2 \leq \lambda_1$ and the coefficients $A_j$, $B_j$, $C_j$ depended on two arbitrary constants. Euler also considered the case in which the roots of $\lambda(\lambda - 1)a + \lambda c + f = 0$ were complex (cf. Euler [1768–1770, 2: §.985]).

***

In chapter X, section 1, book 2, Euler attempted to represent the solution $y(u)$ of the equation

$$(3.12) \quad L(u) \frac{d^2 y}{du^2} + M(u) \frac{dy}{du} + N(u) y = U(u),$$

where $L(u)$, $M(u)$, $N(u)$, and $U(u)$ are functions of $u$, in terms of a definite integral. He tackled the question by considering a definite integral of the type

$$(3.13) \quad y = \int_{x_1}^{x_2} V(x, u) dx,$$

where $V(x, u)$ was a given function. He supposed that (3.13) was a solution to the differential equation (3.12) and sought to determine the form of the coefficients $L(u)$, $M(u)$, $N(u)$, and $U(u)$ under this hypothesis$^{59}$. For example, Euler considered

$$(3.14) \quad y = \int_0^a e^{K(u)Q(x)} P(x) dx,$$  

$^{58}$In this case the coefficients $\beta_j$ became equal to infinity for one of the two roots.

$^{59}$In modern terms, the integrals $y = \int_{x_1}^{x_2} V(x, u) dx$ can be viewed as integral transforms (see Deakin [1981] and [1985]). An integral transform is a map $F(u) = \int_{x_1}^{x_2} K(x, u) f(x) dx$, where $x_1$ and $x_2$ are constants, which transforms a function $f(x)$ into another function $F(u)$. An example is the Laplace transform: $F(u) = \int_0^\infty e^{-ux} f(x) dx$, provided the integral converges.

$^{60}$Euler wrote $y = \int e^{KQ} P dx$ and specified that the integral vanishes for $x = b$ and gives the value of $y$ for $x = a$. 
where $K(u)$, $Q(x)$ and $P(x)$ were appropriate functions, and calculated the derivatives

$$\frac{dy}{du} = \int_b^a e^{K(u)Q(x)} P(x)K'(u)Q(x)dx$$

and

$$\frac{d^2y}{du^2} = \int_b^a e^{K(u)Q(x)} P(x) [K''(u)Q(x) + (K'(u)Q(x))^2]dx.$$  \hspace{1cm} (3.15)

By substituting (3.14) and (3.15) into (3.12), he obtained

$$L \frac{d^2y}{du^2} + M \frac{dy}{du} + Ny = \int_b^a e^{KQ} P[N + MK'Q + LK''Q + L(K'Q)^2]dx.$$  \hspace{1cm} (3.16)

Then Euler put

$$\int_b^a e^{KQ} P[N + MK'Q + LK''Q + L(K'Q)^2]dx = e^{K(x)Q(x)} R(x) + \gamma,$$

where $\gamma$ was a constant to be determined under the condition $e^{K(u)Q(b)} R(b) + \gamma = 0$. By differentiating equation (3.16), he derived

$$dR + K RdQ = Pdx[N + (MK' + LK'')Q + L(K'Q)^2].$$

At this point he put $L(K')^2 = A + aK$, $MK' + LK'' = B + bK$, $N = C + cK$, where $A$, $B$, $C$, $a$, $b$, $c$ were constants, to obtain the pair of equations:

$$dR = Pdx(C + BQ + AQ^2)$$  \hspace{1cm} (3.17)

and

$$RdQ = Pdx(c + bQ + aQ^2).$$  \hspace{1cm} (3.18)

Hence $\frac{dR}{R} = dQ \frac{C + BQ + AQ^2}{c + bQ + aQ^2}$. The integration of this equation provided the function $R$. It followed immediately from (3.18) that $Pdx = \frac{RdQ}{c + bQ + aQ^2}$ and

$$y = \int_b^a e^{K(u)Q(x)} \frac{RdQ}{c + bQ + aQ^2}.$$  \hspace{1cm} (3.19)

Equation (3.19) gave the solution to the differential equation (3.12) if $U(u)$ was equal to $e^{K(u)Q(u)} R(a)$ (cf. [1768–1770, 2: §1053]).
In a similar way Euler solved other equations of the type (3.12). For instance, he showed that the transformations

\[ y = \int_0^c x^n \sqrt{u^2 + x^2} \, dx, \quad y = \int_0^c x^{n-1} (u^2 + x^2)^\mu (c^2 - x^2)^\nu \, dx \]

and

\[ y = \int_0^c e^{ux} x^n (c - x)^\nu \, dx \]

solved the differential equations

\[ u(c^2 + u^2) \frac{d^2 y}{dx^2} - (n + 1)(c^2 + u^2) \frac{dy}{dx} y + (n + 1) u y \frac{du}{dx} = 0, \]

\[ u(c^2 + u^2) \frac{d^2 y}{dx^2} - (n + 2\mu - 1)(c^2 + u^2) \frac{dy}{dx} y - 2(\mu + \nu) u^2 \frac{dy}{dx} y + 2\mu(n + 2\mu + 2\nu) u y \frac{du}{dx} y = 0, \]

\[ \frac{u d^2 y}{dx^2} + (n + \nu + 2 - au) \frac{dy}{du} - (n + 1) ay = 0, \]

respectively [1768-1770, 2: §. 1026–1033].

§3.3 Functions of more than one variable

The investigation of functions of two or more variables was one of the main novelties of Euler’s analysis. In the Calculus differentialis Euler dealt with what today is named partial derivation. He first defined the differential of a function \( V(x, y, z) \) in the variables \( x, y, z \) by letting \( dV = V(x + dx, y + dy, z + dz) - V(x, y, z) \). By analysing two examples, he observed that \( dV \) could be expressed as \( dV = pdx + qdy + rdz \) where \( p, q, r \) are functions of

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61 Euler gave the following definition of a function of two or more variables:

Even though we have so far examined more than one variable quantity, they were connected so that each of them was the function of only one variable and once the value of one variable was determined, the others would be simultaneously determined at the same time. We shall now consider certain variable quantities that do not depend on one another; if a determined value is given to one of these variables, the others remain indeterminate and variable. It would be convenient to denote such variables with \( x, y, z \), because they comprise all determined values; if they are compared with each other, they will be completely unconnected, since it is legitimate to replace any value of one of them such as \( z \), and the others, \( x \) and \( y \), remain entirely free as before. This is the difference between dependent variable quantities and independent variable quantities. In the first case, if we determine one, all the others are determined. In the second case, the determination of a variable in no way restricts the meanings of the others.

[...] Therefore a function of two or more variable quantities \( x, y, z \) is an expression composed of these quantities in whatever manner [1748, 1: §§77–78].
x, y, z. He also noted that if y and z were taken as constants then \(dy = 0\), \(dz = 0\), and \(dV = pdx\). Similarly, if x and y were constants then \(dx = 0\), \(dy = 0\), and \(dV = rdx\); if x and z were constants, then \(dx = 0\), \(dz = 0\) and \(dV = qdy\). Consequently, \(dV\) was obtained by calculating the differentials of V supposing, on each occasion, that two of the variables were constant [1755, §§208–213].

Euler then demonstrated the theorem of mixed differentials which, in the case of functions with two variables, could be formulated as follows\(^{52}\):

If \(dV = Pdx + Qdy\) then the differential of \(P\) for variable \(y\) and constant \(x\) and the differential of \(Q\) for variable \(x\) and constant \(y\) are equal.

The proof runs as follows (cf. [1755, §§226–228]). Consider a function \(V\) of the variables \(x\) and \(y\) and put \(A = V(x, y)\), \(B = V(x + dx, y)\), \(C = V(x, y + dy)\), \(D = V(x + dx, y + dy)\). Take the differential of \(V\), holding \(x\) constant: this produces \(C - A = Qdy\). If in \(C - A\) we put \(x + dx\) in place of \(x\), it produces \(D - B\), the differential of which (namely, the differential of \(Qdy\) for variable \(x\)) is

\[
D - B - C + A.
\]

Now, if \(x + dx\) is put into \(A\) in place of \(x\), then \(B\) is produced, and then the differential of \(A\), taking \(x\) to be the variable, is \(B - A (= Pdx)\). Putting \(y + dy\) in place of \(y\) in \(B - A\) we obtain the differential of \(Pdx\) for variable \(y\):

\[
D - B - C + A.
\]

Since this differential is equal to the differential found in the previous operations, the theorem is proved.

Subsequently, Euler posed \(dP = rdy\) (constant \(x\)) and \(dQ = qdx\) (constant \(y\)), and observed that \(dPdx = rdxdy\) and \(dQdx = qdx dy\). Since the mixed differentials are equal, he had \(r = q\). At this point Euler decided to introduce a symbolism to indicate the functions \(r\) and \(q\) in a convenient and unambiguous way. He denoted \(r\) by means of the symbol \((\frac{dP}{dy})\), which meant the differential of \(P\) for variable \(y\) and constant \(x\) (that is, considering \(P\) as a function of the single variable \(y\)) divided by \(dy\). Similarly \((\frac{dQ}{dx})\) indicated the differential of \(Q\) for variable \(x\) and constant \(y\). Therefore the condition that linked the finite quantities \(P\) and \(Q\) in the differential \(dV = Pdx + Qdy\) can be expressed as \((\frac{dP}{dy}) = (\frac{dQ}{dx})\) [1755, §§231–232].

\(^{52}\)Euler first published this proposition in his [1734–35]. On this theorem, see Fraser [1889, 319–321].
In the third book of the *Calculus integralis*, Euler dealt with the integration of functions of two variables. In his [1765] he had stated that the integration of a function of two variables is a new and as little developed field of the integral calculus, which "differs very much from the common integral calculus, where functions of a variable only occur. It demands entirely special rules, even if it also used the devices of the first part [of calculus]" [1765, 20].

In the *Calculus integralis*, Euler explained that if one integrates a function \( X(x) \) of one variable \( x \), one obtains \( \int X(x)dx = F(x) + C \), where \( F(x) \) is a function such that \( \frac{dF(x)}{dx} = X(x) \) and \( C \) is an arbitrary constant. In the same way, if one integrates a function \( Z(x,y) \) of the variables \( x \) and \( y \) with respect to \( x \), one obtains \( \int Z(x,y)dx = F(x,y) + f(y) \), where \( F(x,y) \) is a function such that \( \frac{dF(x,y)}{dx} = Z(x,y)dx \) and \( f(y) \) is an arbitrary quantity dependent on \( y \) [1768–70, 3: §§39–40]. For instance, if \( \frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} \), then \( Z(x,y) = \sqrt{x^2+y^2} + f(y) \) [1768–70, 3:§47].

According to Euler, it was not necessary that the quantity \( f(y) \) was one of the functions accepted in analysis, which were given by means of one only analytical expressions. Indeed \( f(y) \) could be thought of as the ordinate of a curve whose abscissa is \( y \), which could be traced by a free stroke of the hand or composed of pieces of more than one continuous curve [1768–70, 3: §37]. Therefore Euler termed these quantities as discontinuous functions. In opposition functions composed by one only analytical formula were termed continuous functions. In this way the notion of discontinuity, which had previously been considered only in reference to curves (see section 1.3), was extended to functions.

However Euler did not developed a satisfactory analytical theory of discontinuous functions. Thus, to explain what was the differential ratio of a discontinuous function, Euler merely used the geometric meaning of a function and stated that if \( f(x) \) represented a curve, then the differential ratio \( f'(x) \) was the slope of the tangent whereas, if \( f(x) \) was interpreted as an area, then \( f'(x) \) was a curve [1768–70, 3: §§95–96]. This geometrical interpretation was problematic for mathematicians such as Euler who wanted to separate analysis from geometry. Euler could avoid it only if he had restructured analysis and changed the notion of a function and its formal methodology; but this did not happen. Discontinuous functions were never
really manipulated and computed and so they were not considered functions in the proper sense of the term. They remained a marginal addition to the calculus of old continuous functions: they were useful in the interpretation of the solutions to partial differential equations but did not really enter into the procedures that enable the determination of these solutions. In conclusion I will give two examples of Euler’s treatment of partial differential equations. First, take the equation

\[ \frac{\partial^2 z}{\partial x^2} = P(x, y). \]

Euler solved it by integrating twice with respect to \( x \). He first obtained \( \frac{\partial z}{\partial x} = \int P(x, y)dx + f(y) \) and then

\[ z = \int dx \int P(x, y)dx + xf(y) + F(y), \]

where \( f(y) \) and \( F(y) \) could also be interpreted as discontinuous functions of \( y \) (cf. [1768-70, 3: §8.245–248]).

Second, consider the wave equation \( \frac{\partial^2 z}{\partial y^2} = \alpha^2 \frac{\partial^2 z}{\partial x^2} \). By the change of variables \( t = x + ay, u = x - ay \), Euler derived \( \frac{\partial z}{\partial t} = 0 \). He then integrated with respect to \( t \) so to obtain \( \frac{\partial z}{\partial u} = h(u) \). Hence \( z = \int h(u)du + f(t) = F(u) + f(t) \) and

\[ z = f(x + ay) + F(x - ay). \]

According to Euler, the functions \( f \) and \( F \) could be discontinuous functions [1768–70, 3: §296].

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In [1768–70, 3: §299] Euler also illustrated d’Alembert’s solution to the wave equation.


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Giovanni Ferraro
Università del Molise
Dipartimento STAT,
c.da Fonte Lappone
86090 Pesche (Isernia) Italy
giovanni.ferraro@unimol.it